# $\mathcal{H}_\infty$ observer design for Singular Nonlinear Parameter-varying System

Journée du Groupe de Travail "Sûreté - Surveillance - Supervision"


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# Outline

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   - Unknown input observer-based Approach
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# Contents

# DO Manh-Hung

- Hochiminh city, Vietnam.
- Engineering Degree in Automation (ESISAR - Grenoble INP)
- Internship and PhD Thesis at GIPSA-lab (Grenoble, 10/2017 - 09/2020)
- PhD Topic: Fault tolerant control for LPV (or switched) system with application to the control of the vertical dynamics of a vehicle.
- Grant 15016 EMPHYSIS – ITEA3 European Project.
- Platform INOVE: Racing car model with nonlinear Electro-Rheological damper


$\Longrightarrow$ Displacement Sensors, Accelerometers, Matlab/DSpace Interface.

## Journal papers

- *Observer-based fault tolerant control design for a class of LPV descriptor systems*. Journal of the Franklin Institute 2019.
- *Robust observer-based controller for uncertain stochastic LPV system under actuator degradation*. International Journal of Robust and Nonlinear Control, under second revision.
- $\mathcal{H}_\infty$ *observer design for singular nonlinear parameter-varying system*. IEEE Control Systems Letters + IEEE CDC, under review .
- *Unknown input observer design for singular nonlinear parameter-varying system with time-varying delay*. IEEE Transactions on Automatic Control, under review .

## Conference papers

- *An integrated design for robust actuator fault accommodation based on* $\mathcal{H}_\infty$ *proportional-integral observer*. IEEE Conference on Decision and Control 2018.
- *Robust* $\mathcal{H}_\infty$ *proportional integral observer design for actuator fault estimation*. International Mini Conference on Vehicle System Dynamics, Identification and Anomalies, VSDIA 2018.
- *Robust* $\mathcal{H}_\infty$ *proportional-integral observer for fault diagnosis: Application to vehicle suspension*.IFAC SAFEPROCESS 2018.
- *Robust* $\mathcal{H}_2$ *observer design for actuator degradation: Application to suspension system*. Conference on Control and Fault Tolerant Systems (SysTol) 2019.
- *Fault estimation methods in descriptor system with partially decoupled disturbances*. IFAC World Congress 2020, accepted.
- *Frequency-shaping observer-based controller design for actuator degradation : Application to suspension system*. Mediterranean Conference on Control and Automation (MED) 2020. IEEE, under review.

# Contents

**Linear Time-Varying (LPV) system** $\rightarrow$ Modeling the nonlinear system:

$$\text{Nonlinear System} \overset{\rho \in \mathcal{P}_\rho}{\Longleftrightarrow} \begin{cases} \dot{x} &= A_{(\rho)}x + B_{(\rho)}u + W_{1(\rho)}w \\ y &= C_{y(\rho)}x + D_{(\rho)}u + W_{2(\rho)}w \end{cases} \tag{1}$$

where $x$ is system state, $u$ is control input, $w$ is disturbance vector, $\rho$ is measured or estimated time-varying parameter, and parameter space $\mathcal{P}_\rho$:

$$\mathcal{P}_\rho = \{\rho = \begin{bmatrix}\rho_1(t) & \rho_2(t) & \dots & \rho_p(t)\end{bmatrix}^T | \underline{\rho_i}(t) \leq \rho_i \leq \bar{\rho_i}(t)\}, \quad \forall\, i = 1:p,\ t \geq 0.$$

**Nonlinear Parameter-Varying System (NLPV)** $\rightarrow$ Handling $\rho$ is nonlinear function

- Boulkroune et al. [2015] $\rightarrow$ Observer for the NLPV diesel engines with once differentiable nonlinearity.
- us Saqib et al. [2017], Yang et al. [2019] $\rightarrow \mathcal{H}_\infty$ output/state-feedback controller for NLPV model.
- Pham et al. [2019] $\rightarrow \mathcal{H}_\infty$ Luenberger observer for Suspension system with non-linear damper force.

$\rightarrow$ Parameter-independent stability ($V = e^T Pe$, $P$ is constant) which may narrow the feasible solution region.

**Singular LPV system** → LPV + Static Constraints

$$\text{Nonlinear System} \overset{\rho\in\mathcal{P}_\rho}{\Longleftrightarrow} \begin{cases} E\dot{x} &= A_{(\rho)}x + B_{(\rho)}u + W_{1(\rho)}w \\ y &= C_{y(\rho)}x + D_{(\rho)}u + W_{2(\rho)}w \end{cases} \tag{2}$$

$$\Leftrightarrow \underbrace{\begin{cases} \dot{x}_1 &= A_{1(\rho)}x_1 + B_{1(\rho)}u + W_{11(\rho)}w \\ y_1 &= C_{1(\rho)}x + D_{1(\rho)}u + W_{21(\rho)}w \end{cases}}_{\textbf{SLOW SUBSYSTEM}} + \underbrace{\begin{cases} N\dot{x}_2 &= x_2 + B_{1(\rho)}u + W_{12(\rho)}w \\ y_2 &= C_{2(\rho)}x + D_{2(\rho)}u + W_{22(\rho)}w \end{cases}}_{\textbf{FAST SUBSYSTEM}}$$

where $N$ is nilpotent and $y = y_1 + y_2$.

- Hamdi et al. [2012] → UI PI observer with decoupling constraint.
- Rodrigues et al. [2014] → Fast adaptive estimation observer
- Shi and Patton [2015] → $\mathcal{H}_\infty$ proportional-derivative observer (singular observer).
- López-Estrada et al. [2015] → $\mathcal{H}_\infty$ functional observer for uncertainty in $\rho$.

→ Polytopic model where $C_{y(\rho)} = C$ constant, which limits the implementation.

# Contributions of the presentation

1. A new class of singular NLPV with Lipschitz nonlinearity (S-NLPV) is introduced, which unifies all so far existing kinds of LPV systems;
2. A $\mathcal{H}_\infty$ observer design-based process for the S-NLPV system is studied. In which, both disturbance attenuation and parameter-dependent stability are ensured by solving LMI optimization under the Lipschitz constraint.

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## S-NLPV system

$$\begin{cases} E\dot{x} & = A_{(\rho)}x + B_{(\rho)}u + B_{\phi(\rho)}\phi(x,u) + D_{1(\rho)}w \\ y & = C_{y(\rho)}x + D_{2(\rho)}w \\ z & = C_z x \end{cases} \tag{3}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $y \in \mathbb{R}^{n_y}$ is the measurement output vector, $u \in \mathbb{R}^{n_u}$ is the input vector, $w \in \mathbb{R}^{n_d}$ is the disturbance vector with bounded energy, and $z \in \mathbb{R}^{n_z}$ is the vector of desired signals, which is a combination of $x$, to be estimated ($C_z = I \rightarrow$ State-feedback controller; $C_z \neq I \rightarrow$Output-feedback controller).

## Assumptions

(A.1) Parameter variations are bounded. In other words, $|\dot{\rho_i}| \leq \vartheta_i$ where $\vartheta_i$ is non-negative constant boundness Wu et al. [1996].

(A.2) Nonlinear term $\phi(x,u)$ with $u$ bounded (due to saturation in practice) is Lipschitz function satisfying:

$$||\tilde{\phi}|| = ||\phi(x,u) - \phi(\hat{x},u)|| \leq \gamma ||x - \hat{x}|| \tag{4}$$

for all $x, \hat{x} \in \mathbb{R}^{n_x}$, where $\gamma$ is known Lipschitz constant.

(A.3) S-NLPV system (3) is R-detectable (for slow subsystem) and impulse-free (for fast subsystem) $\forall \rho$, which is analytically verified by the conditions discussed later.

# Objectives

$\Longrightarrow$ Determine a $\mathcal{H}_\infty$ observer has the form:

## NLPV observer

$$\begin{cases} \dot{\xi} &= F_{(\rho)}\xi + J_{(\rho)}u + L_{(\rho)}y + T_{(\rho)}B_{\phi(\rho)}\phi(\hat{x},u) \\ \hat{x} &= \xi + N_{(\rho)}y \\ \hat{z} &= C_z\hat{x} \end{cases}, \tag{5}$$

In which, $\hat{z}$ is estimated state of $z$ and the observer matrices $F_{(\rho)}$, $J_{(\rho)}$, $T_{(\rho)}$, $L_{(\rho)}$, and $N_{(\rho)}$ are synthesized later.

## Design Objectives

(O.1) When $w^* = \begin{bmatrix} w^T & \dot{w}^T \end{bmatrix}^T = 0$ (explained later), estimation error dynamics are asymptotically stable.

(O.2) When $w^* \neq 0$, the impact of disturbance $w^*$ on the desired estimation error $e_z = z - \hat{z}$ is attenuated, i.e.

$$\sup_{\rho \in \mathcal{P}_\rho} \sup_{\|w^*\|_2 \neq 0, w^* \in \mathcal{L}_2} \frac{\|e_z\|_2}{\|w^*\|_2} \leq \gamma_\infty. \tag{6}$$

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Define the state estimation error $e = x - \hat{x}$:

$$e = T_{(\rho)}Ex - \xi - N_{(\rho)}D_{2(\rho)}w, \tag{7}$$

where $T_{(\rho)}$ has to ensure the constraint:

$$T_{(\rho)}E + N_{(\rho)}C_{y(\rho)} = I. \tag{8}$$

Hence, the error dynamics is displayed as:

$$\begin{aligned}\dot{e} = {} & F_{(\rho)}e + T_{(\rho)}B_{\phi(\rho)}\tilde{\phi} + (J_{(\rho)} - T_{(\rho)}B_{(\rho)})u \\ & + (T_{(\rho)}A_{(\rho)} - F_{(\rho)}T_{(\rho)}E - L_{(\rho)}C_{y(\rho)})x - N_{(\rho)}D_{2(\rho)}\dot{w} \\ & + [T_{(\rho)}D_{1(\rho)} + (F_{(\rho)}N_{(\rho)} - L_{(\rho)})D_{2(\rho)}]w\end{aligned} \tag{9}$$

**Conditions $\rightarrow$ Handle the coupling between $e$, $x$, and $u$**

$$J_{(\rho)} - T_{(\rho)}B_{(\rho)} = 0, \tag{10}$$

$$T_{(\rho)}A_{(\rho)} - F_{(\rho)}T_{(\rho)}E - L_{(\rho)}C_{y(\rho)} = 0, \tag{11}$$

$$K_{(\rho)} = -F_{(\rho)}N_{(\rho)} + L_{(\rho)}, \tag{12}$$

Thereby, the dynamics (9) is rewritten as:

$$\dot{e} = F_{(\rho)}e + T_{(\rho)}B_{\phi(\rho)}\tilde{\phi} + \left[(T_{(\rho)}D_{1(\rho)} - K_{(\rho)}D_{2(\rho)}) \quad -N_{(\rho)}D_{2(\rho)}\right] w^{*}. \tag{13}$$

$\Longrightarrow$ $e$ depends on the nonlinear term $\tilde{\phi}$ and the disturbance $w^{*} = \left[w^T \quad \dot{w}^T\right]^T$.

**Decoupling condition between $e$ and $w^*$**

$$\begin{cases} T_{(\rho)}D_{1(\rho)} - K_{(\rho)}D_{2(\rho)} = 0, \\ N_{(\rho)}D_{2(\rho)} = 0. \end{cases} \tag{14}$$

$\rightarrow$ Restrictive condition.
$\rightarrow$ Unknown input observer design (discussed later).

If (14) is not satisfied
$\rightarrow$ $\mathcal{H}_\infty$ observer design (main result of this representation)

$\rightarrow$ Design process:
From Eqs. (8), (11) and (12), it follows that:

$$T_{(\rho)}A_{(\rho)} - K_{(\rho)}C_{y(\rho)} - F_{(\rho)} = 0. \tag{15}$$

From conditions (8) and (15), we obtain:

$$\begin{bmatrix} T_{(\rho)} & N_{(\rho)} & F_{(\rho)} & K_{(\rho)} \end{bmatrix} \underbrace{\begin{bmatrix} E & A_{(\rho)} \\ C_{y(\rho)} & 0 \\ 0 & -I \\ 0 & -C_{y(\rho)} \end{bmatrix}}_{\theta_{(\rho)}} = \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{\psi} \tag{16}$$

If $\mathrm{rank}\begin{bmatrix}\theta_{(\rho)}\\ \psi\end{bmatrix} = \mathrm{rank}(\theta_{(\rho)}) \Leftrightarrow \mathrm{rank}(\theta_{(\rho)}) = 2n_x$ (Koenig [2005])

**General Solution → Derived from $\psi$ and Moore-Penrose inverse of $\theta_{(\rho)}$**

$$\begin{bmatrix} T_{(\rho)} & N_{(\rho)} & F_{(\rho)} & K_{(\rho)} \end{bmatrix} = \psi\theta^{\dagger}_{(\rho)} - Z_{(\rho)}(I - \theta_{(\rho)}\theta^{\dagger}_{(\rho)}) = \Gamma_{1(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)} \quad (17)$$

where $\Gamma_{1(\rho)} = \psi\theta^{\dagger}_{(\rho)}$, $\Gamma_{2(\rho)} = I - \theta_{(\rho)}\theta^{\dagger}_{(\rho)}$, and $Z_{(\rho)}$ is a parameter-dependent arbitrary matrix calculated later.
It follows that:

$$T_{(\rho)} = (\Gamma_{1(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)})\delta_T, \quad (18)$$
$$N_{(\rho)} = (\Gamma_{1(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)})\delta_N, \quad (19)$$
$$F_{(\rho)} = (\Gamma_{1(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)})\delta_F, \quad (20)$$
$$K_{(\rho)} = (\Gamma_{1(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)})\delta_K, \quad (21)$$

where:

$$\delta_T = \begin{bmatrix} I\\0\\0\\0\end{bmatrix}, \delta_N = \begin{bmatrix} 0\\I\\0\\0\end{bmatrix}, \delta_F = \begin{bmatrix} 0\\0\\I\\0\end{bmatrix}, \delta_K = \begin{bmatrix} 0\\0\\0\\I\end{bmatrix}. \quad (22)$$

## Error Dynamics

$$\begin{cases} \dot{e} &= F_{(\rho)}e + B_{e(\rho)}\tilde{\phi} + W_{(\rho)}w^*, \\ e_z &= C_{z(\rho)}e \end{cases} \tag{23}$$

where $e_z = z - \hat{z} = C_{z(\rho)}(x - \hat{x}) = C_z e$, $\delta_{\phi(\rho)} = \delta_T B_{\phi(\rho)}$; $\delta_{1(\rho)} = \delta_T D_{1(\rho)}$, $\delta_{2(\rho)} = \delta_K D_{2(\rho)}$, $\delta_{3(\rho)} = \delta_N D_{2(\rho)}$, and

$$B_{e(\rho)} = \Gamma_{1(\rho)}\delta_{\phi(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)}\delta_{\phi(\rho)}, \tag{24}$$

$$W_{(\rho)} = \begin{bmatrix} W_{1(\rho)} & W_{2(\rho)} \end{bmatrix}, \tag{25}$$

$$W_{1(\rho)} = \Gamma_{1(\rho)}(\delta_1 - \delta_2) - Z_{(\rho)}\Gamma_{2(\rho)}(\delta_1 - \delta_2), \tag{26}$$

$$W_{2(\rho)} = -(\Gamma_{1(\rho)}\delta_{3(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)}\delta_{3(\rho)}), \tag{27}$$

## Design Objectives $\rightarrow$ Find $Z_{(\rho)}$ such that:

(O.1) When $w^* = 0$, error dynamics are asymtotically stable.

(O.2) When $w^* \neq 0$, the impact of disturbance $w^*$ on the desired estimation error $e_z = z - \hat{z}$ is attenuated, i.e.

$$\sup_{\rho \in \mathcal{P}_\rho} \sup_{\|w^*\|_2 \neq 0, w^* \in \mathcal{L}_2} \frac{\|e_z\|_2}{\|w^*\|_2} \leq \gamma_\infty. \tag{28}$$

# Design Solution

## Theorem 1.

Under the assumptions (A.1)-(A.3), the design objectives (O.1)-(O.2) are achieved if there exist symmetric positive definite matrices $P_{(\rho)}$ and matrix $Y_{(\rho)}$, positive scalar $\epsilon$ which minimize $\gamma_\infty$ and satisfy that:

$$\begin{bmatrix} \Omega_{11(\rho)}+\eta & \Omega_{12(\rho)} & \Omega_{13(\rho)} & \Omega_{14(\rho)} & C_z^T \\ (*) & -\epsilon I & 0 & 0 & 0 \\ (*) & (*) & -\gamma_\infty I & 0 & 0 \\ (*) & (*) & (*) & -\gamma_\infty I & 0 \\ (*) & (*) & (*) & (*) & -I \end{bmatrix} < 0, \tag{29}$$

then the matrix $Z_{(\rho)}$ is calculated by: $Z_{(\rho)} = -P_{(\rho)}^{-1} Y_{(\rho)}$.

$$\Omega_{11(\rho)} = \sum_i^p \pm\vartheta_i \frac{\partial P_{(\rho)}}{\partial \rho_i} + \mathcal{H}\{P_{(\rho)}\Gamma_{1(\rho)}\delta_F + Y_{(\rho)}\Gamma_{2(\rho)}\delta_F\}, \tag{30}$$

$$\Omega_{12(\rho)} = P_{(\rho)}\Gamma_{1(\rho)}\delta_\phi + Y_{(\rho)}\Gamma_{2(\rho)}\delta_\phi, \tag{31}$$

(32)

$$\Omega_{13(\rho)} = P_{(\rho)}\Gamma_{1(\rho)}(\delta_1 - \delta_2) + Y_{(\rho)}\Gamma_{2(\rho)}(\delta_1 - \delta_2), \tag{33}$$

$$\Omega_{14(\rho)} = -(P_{(\rho)}\Gamma_{1(\rho)}\delta_3 + Y_{(\rho)}\Gamma_{2(\rho)}\delta_3), \tag{34}$$

$$\eta = \epsilon(\gamma I)^T(\gamma I), \tag{35}$$

*Remark 1:* The notion $\sum_i^p \pm(.)$ expresses all combinations of $+(.)$ and $-(.)$ that should be included in the inequality (29). Consequently, the inequality (29) actually represents $2^p$ different inequalities that correspond to the $2^p$ different combinations in the summation.

*Proof:* Proof is demonstrated later.

From $Z_{(\rho)}$ obtained in Theorem 1 $\rightarrow$ Observer Matrices:

$$T_{(\rho)} = (\Gamma_{1(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)})\delta_T, \tag{36}$$

$$N_{(\rho)} = (\Gamma_{1(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)})\delta_N, \tag{37}$$

$$F_{(\rho)} = (\Gamma_{1(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)})\delta_F, \tag{38}$$

$$K_{(\rho)} = (\Gamma_{1(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)})\delta_K, \tag{39}$$

$$J_{(\rho)} = T_{(\rho)}B_{(\rho)}, \tag{40}$$

$$L_{(\rho)} = K_{(\rho)} + F_{(\rho)}N_{(\rho)}. \tag{41}$$

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# Numerical Model

## S-NLPV system

$$\begin{cases} E\dot{x} & = A_{(\rho)}x + Bu + B_\phi sin(Kx)u + D_1 w \\ y & = C_{y(\rho)}x + D_2 w \\ z & = C_z x \end{cases} \tag{42}$$

- Desired signal $z = \begin{bmatrix} z_1^T & z_2^T & z_3^T \end{bmatrix}^T$ is the output vector to be estimated.
- Varying-parameter $\rho$ are defined as: $\rho = 0.25sin(8t) + 0.75$
- System parameters are chosen as following:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_{(\rho)} = \begin{bmatrix} -5+\rho & 1 & 1 \\ 0 & -5 & 0 \\ 0.5 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0.2 \\ 0.5 \end{bmatrix}, B_\phi = \begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix}, C_{y(\rho)} = \begin{bmatrix} 1 & 1 & 0.2\rho \\ 0 & 2 & -1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.5 \\ 0.1 \\ 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, C_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

  and $K = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, which satisfy the conditions (C.1) and (C.2).
- Control input $u$ is bounded in region $|u| \leq u_0 = 5 \rightarrow$ Lipschitz condition:

$$||\phi(x,u) - \phi(\hat{x},u)|| \leq u_0 K ||x - \hat{x}||, \tag{43}$$

where $\phi(x,u) = sin(Kx)u$ and $\gamma = u_0 K$.

# $\mathcal{H}_\infty$ observer design

- **Grid-based solution** $\rightarrow$ Solving Theorem 1

$$\text{Singular LPV} \xLeftrightarrow{\text{Time-frozen } \rho^j} \text{Singular Linear Time-Invariant (LTI)} \tag{44}$$

where $j = 1 : N$, $N = n_g^{\rho_1} \times n_g^{\rho_2} \times \ldots n_g^{\rho_p}$, and $n_g^{\rho_i}$, expressing the number of gridding points for element $\rho_i$ of vector $\rho$.

  - **Basic functions:** The matrices $P_{(\rho)}$ and $Y_{(\rho)}$ are chosen as polynomial functions of $\rho$:

$$P_{(\rho)} = P_0 + \rho P_1 + \rho^2 P_2, \tag{45}$$

$$Y_{(\rho)} = Y_0 + \rho Y_1 + \rho^2 Y_2, \tag{46}$$

  $P_0, P_1, P_2, Y_0, Y_1$, and $Y_2$ are constant matrices found later by Theorem 1.
  - **Number of gridding points:** $p = 1$ (1 element of $\rho$) and $n_g = 20$ points, so $n_g$ values $\rho^j$ $(j = 1 : n_g)$.

- At each time-frozen values $\rho^j$, LPV system is treated as a LTI system at each $\rho^j$ $\Longrightarrow$ Frequency analysis for $\rho^1$, $\rho^{10}$, and $\rho^{20}$ are presented:

$$S_{e_z w(\rho^j)} = C_{z(\rho^j)}(pI - F_{(\rho^j)})^{-1} W_{1(\rho^j)}, \tag{47}$$

$$S_{e_z \dot{w}(\rho^j)} = C_{z(\rho^j)}(pI - F_{(\rho^j)})^{-1} W_{2(\rho^j)}, \tag{48}$$

$$S_{e_z \tilde{\phi}(\rho^j)} = C_{z(\rho^j)}(pI - F_{(\rho^j)})^{-1} B_{e(\rho^j)}. \tag{49}$$

- Toolbox Yalmip Lofberg [2004] and solver *sdpt3* Toh et al. [1999] for $0.5 \leq \rho \leq 1$ and $|\dot{\rho}| \leq \vartheta = 2$.
- Optimal $\mathcal{H}_\infty$ performance: $\gamma_\infty = 0.0014$ (or $-56.9662$ dB) and $\epsilon = 50.5071$.

# Frequency Analysis: Disturbance $w$ to $e_z$ estimation error

- Sensitivity functions: $|S_{e_z w}| = |e_z|/|w|$. ($e_z = e_x$ as $C_z = I$)

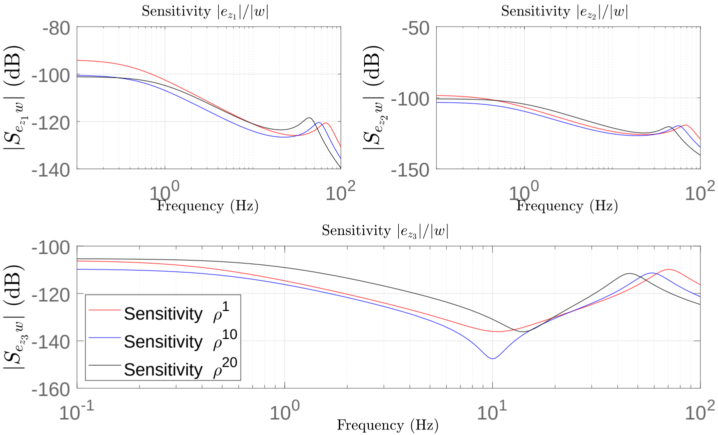


Figure 1: Sensitivity function $|S_{e_z w}| = |e_z/w|$

$$\Longrightarrow |S_{e_z w}| < \gamma_\infty \text{ (-56.9662 dB)}$$

# Frequency Analysis: Disturbance $\dot{w}$ to $e_z$ estimation error

- Sensitivity function: $|S_{e_z\dot{w}}| = |e_z|/|\dot{w}|$. ($e_z = e_x$ as $C_z = I$)

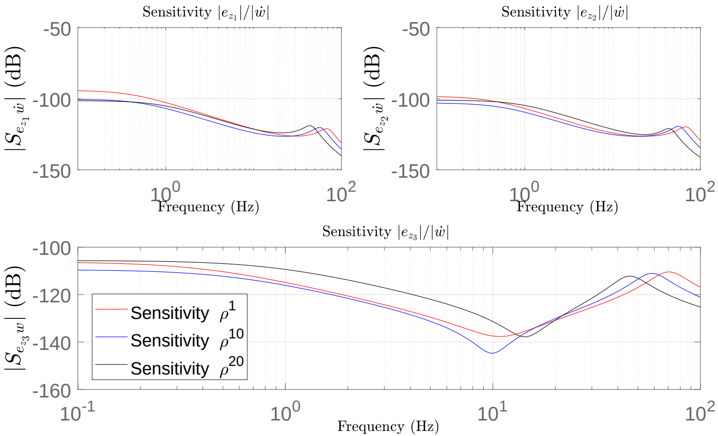


Figure 2: Sensitivity function $|S_{e_z\dot{w}}| = |e_z/\dot{w}|$

$$\Longrightarrow |S_{e_z\dot{w}}| < \gamma_\infty \text{ (-56.9662 dB)}$$

# Frequency Analysis: Nonlinearity to $e_z$ estimation error

- Sensitivity function: $|S_{e_z\tilde{\phi}}| = |e_z|/|\tilde{\phi}|$. ($e_z = e_x$ as $C_z = I$)

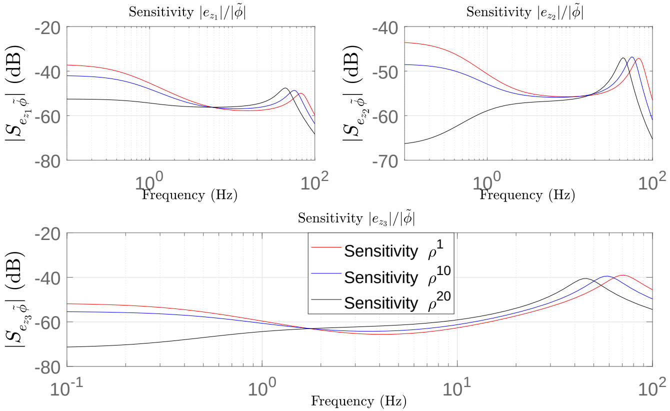


Figure 3: Sensitivity function $|S_{e_z\tilde{\phi}}| = |e_z/\tilde{\phi}|$

$\Longrightarrow$ Difference of nonlinearity term $\tilde{\phi}$ can affect the accuracy of estimation, especially $x_1$ and $x_2$.

# Test Condition

The time-domain simulation is realized with the following conditions:

- Duration: 3 seconds.
- Disturbance vector is defined as:

$$w = sin(4\pi t). \tag{50}$$

- Control input:

$$u = u_0 sin(8\pi t). \tag{51}$$

- Initial condition: $x_1(0) = 0$, $x_2(0) = 0$ and $\hat{x}_0 = \begin{bmatrix} 0.005 & 0 & 0.02 \end{bmatrix}^T$.

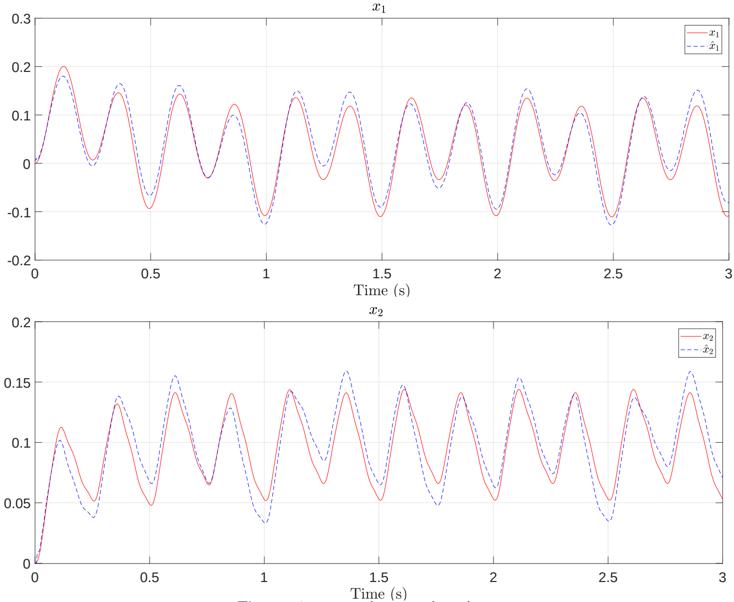


Figure 4: $x_1$ and $x_2$ estimation

$\Longrightarrow$ Impact of $\tilde{\phi}$ on $x_1$ and $x_2$ as mentioned in Frequency Analysis.

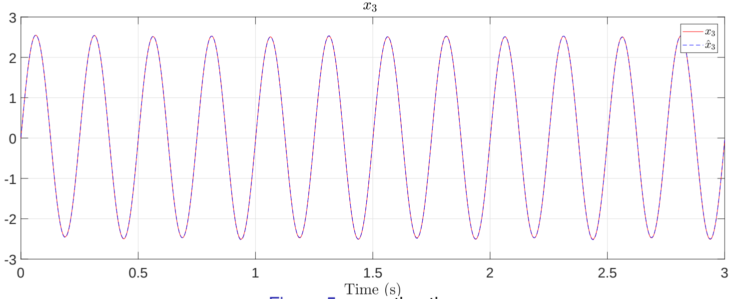


Figure 5: $x_3$ estimation

Table 1: Evaluation for estimation error

| Signal | $e_{x_1}$ | $e_{x_2}$ | $e_{x_3}$ |
|---|---|---|---|
| RMS (Root-mean-square) | 0.0192 | 0.0143 | 0.0071 |

## $H_\infty$ observer's drawbacks

**Problem 1:**

$$y = C_{y(\rho)}x + D_2 w = \begin{bmatrix} 1 & 1 & 0.2\rho \\ 0 & 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} w \quad (52)$$

$\Longrightarrow$ $w$ has a direct transfer to output $y$, so $\mathcal{H}_\infty$ norm is not effective.

**Problem 2:**
The number of signal $z \Longrightarrow$ Effectiveness of optimization $\frac{\|e_z\|_2}{\|w^*\|_2} \leq \gamma_\infty$

$$z = x \longrightarrow C_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{6 optimizations: 3 for } \frac{\|e_z\|_2}{\|w\|_2} \leq \gamma_\infty, \text{3 for } \frac{\|e_z\|_2}{\|\dot{w}\|_2} \leq \gamma_\infty$$

$\Longrightarrow$ $2^{nd}$ Simulation: Verify above problems

$$z = \bar{C}_z x, \bar{C}_z = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \longrightarrow \text{2 optimizations: 1 for } \frac{\|e_z\|_2}{\|w\|_2} \leq \gamma_\infty, \text{1 for } \frac{\|e_z\|_2}{\|\dot{w}\|_2} \leq \gamma_\infty$$

$$\Longrightarrow \begin{bmatrix} y \\ \hat{z} \end{bmatrix} = \begin{bmatrix} C_{y(\rho)} \\ C_z \end{bmatrix} x + \begin{bmatrix} D_2 \\ 0 \end{bmatrix} w \longrightarrow \hat{x} \approx \begin{bmatrix} C_{y(\rho)} \\ C_z \end{bmatrix}^{-1} \begin{bmatrix} y \\ \hat{z} \end{bmatrix} \quad (53)$$

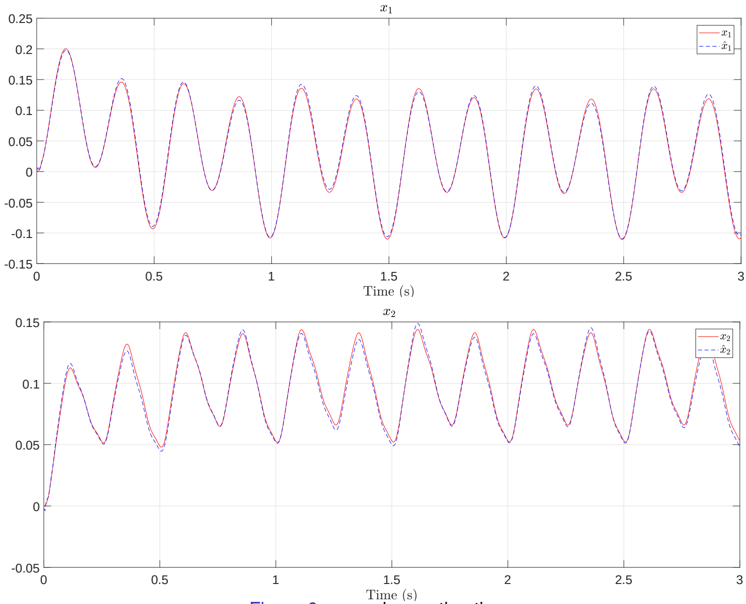


Figure 6: $x_1$ and $x_2$ estimation

$\Longrightarrow$ Better estimation of $x_1$ and $x_2$ compared with $1^{st}$ simulation.

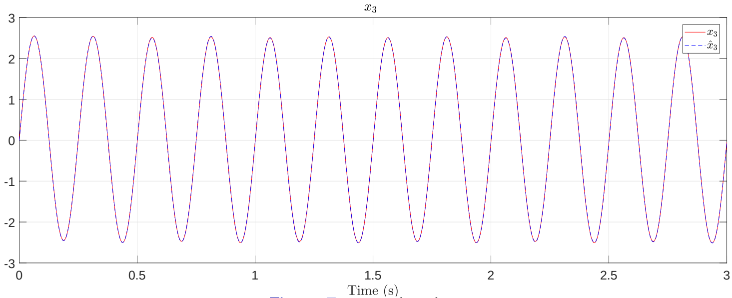


Figure 7: $x_3$ estimation

Table 2: Evaluation for estimation error

| | Simulation 1 | Simulation 2 |
|---|---|---|
| $\gamma_\infty$ | $-56.9691$ dB | $-56.9685$ dB |
| RMS $e_{x_1}$ | $0.0192$ | $0.0039$ |
| RMS $e_{x_2}$ | $0.0143$ | $0.0032$ |
| RMS $e_{x_3}$ | $0.0071$ | $0.0082$ |

# Contents

# Existing Conditions of $\mathcal{H}_\infty$ observer design

**Grid-based solution** $\rightarrow$ Solving Theorem 1

$$\text{Singular LPV} \xLeftrightarrow{\text{Time-frozen } \rho^j} \text{Singular Linear Time-Invariant (LTI)} \tag{54}$$

where $j = 1 : N$, $N = n_g^{\rho_1} \times n_g^{\rho_2} \times \dots n_g^{\rho_p}$.

$\Longrightarrow$ (A.3): "System (3) is R-detectable and impulse-free" is analytically verified at each $\rho^j$ in the grid.

- Stability of error dynamics: $\dot{e} = F_{(\rho)}e + B_{e(\rho)}\tilde{\phi} + W_{(\rho)}w^*$
  If Theorem 1 is feasible
  $\Rightarrow F_{(\rho)} =_{(\rho)} A_{(\rho)} - K_{(\rho)}C_{y(\rho)} = \Gamma_{1(\rho)}\phi_{AC(\rho)} - Z_{(\rho)}\Gamma_{2(\rho)}\phi_{AC(\rho)}$ is Hurwitz.
  $\Rightarrow$ Existence of $Z_{(\rho)}$ to stabilize $F_{(\rho)}$
  $\Rightarrow$ Pair $(\Gamma_{1(\rho)}\phi_{AC(\rho)}, \Gamma_{2(\rho)}\phi_{AC(\rho)})$ is detectable.

$$\Rightarrow \text{rank} \begin{bmatrix} sI - \Gamma_{1(\rho)}\phi_{AC(\rho)} \\ \Gamma_{2(\rho)}\phi_{AC(\rho)} \end{bmatrix} = n_x \forall \rho \Longrightarrow \text{rank} \begin{bmatrix} sE - A_{(\rho^j)} \\ C_{(\rho^j)} \end{bmatrix} = n_x, \quad \forall j = 1 : N, \mathcal{R}(s) \geq 0. \tag{55}$$

  $\Longrightarrow$ R-detectability of slow subsystem in singular system.
- General solution:

$$\text{rank}(\theta_{(\rho)}) = 2n_x \forall \rho \Leftrightarrow \text{rank} \begin{bmatrix} E \\ C_{y(\rho)} \end{bmatrix} = n_x \forall \rho \Longrightarrow \text{rank} \begin{bmatrix} E \\ C_{(\rho^j)} \end{bmatrix} = n_x, \forall j = 1 : N. \tag{56}$$

  $\Longrightarrow$ Impulse-free condition for fast components.

## Impulse-free condition is violated

$\Longrightarrow$ but $\text{rank}\begin{bmatrix} E & B_{\phi(\rho^j)} \end{bmatrix} = \text{rank}(E) = r$ and impulse observable (I-observability)

$$\text{rank}\begin{bmatrix} E & A_{(\rho^j)} \\ 0 & E \\ 0 & C_{(\rho^j)} \end{bmatrix} = n_x + \text{rank}(E) \tag{57}$$

$\Longrightarrow$ Transformation

$$(S.1)\begin{cases} E\dot{x} & = A_{(\rho)}x + B_{(\rho)}u \\ & \quad +B_{\phi(\rho)}\phi(x,u) + D_{1(\rho)}w \\ y & = C_{y(\rho)}x + D_{2(\rho)}w \\ z & = C_z x \end{cases} \overset{(*)}{\Longleftrightarrow} (S.2)\begin{cases} E^*\dot{x} & = A^*_{(\rho)}x + B^*_{(\rho)}u \\ & \quad +B^*_{\phi(\rho)}\phi(x,u) + D^*_{1(\rho)}w \\ y^* & = C^*_{y(\rho)}x + D^*_{2(\rho)}w \\ z & = C_z x \end{cases}$$

(*) Koenig and Mammar [2002]

$$\exists M : M\begin{bmatrix} E & B_{\phi(\rho)} \end{bmatrix} = \begin{bmatrix} E^* & B^*_{\phi(\rho)} \\ 0 & 0 \end{bmatrix}, \text{rank}(E^*) = r, MA_{(\rho)} = \begin{bmatrix} A^*_{(\rho)} \\ A^*_{1(\rho)} \end{bmatrix},$$

$$MB_{(\rho)} = \begin{bmatrix} B^*_{(\rho)} \\ B^*_{1(\rho)} \end{bmatrix}, MD_{1(\rho)} = \begin{bmatrix} D^*_{1(\rho)} \\ D^*_{11(\rho)} \end{bmatrix}, y^* = \begin{bmatrix} -B^*_{1(\rho)}u \\ y \end{bmatrix} = \begin{bmatrix} A_{1(\rho)} \\ C_{y(\rho)} \end{bmatrix} x + \begin{bmatrix} D^*_{11(\rho)} \\ D_{2(\rho)} \end{bmatrix} w$$

$\Longrightarrow$ (S.2) is R-detectable and impulse-observable (impulse observability = condition for general solution in observer design).

$\Longrightarrow$ $\mathcal{H}_\infty$ NLPV observer design (5) with $u$ and $y^*$.

## Unknown input observer-based Approach

Error Dynamics (13) is recalled

$$\dot{e} = F_{(\rho)}e + T_{(\rho)}B_{\phi(\rho)}\tilde{\phi} + \left[(T_{(\rho)}D_{1(\rho)} - K_{(\rho)}D_{2(\rho)}) \quad -N_{(\rho)}D_{2(\rho)}\right] w^*.$$

**Decoupling condition between $e$ and $w^*$**

$$T_{(\rho)}D_{1(\rho)} - K_{(\rho)}D_{2(\rho)} = 0, \tag{58}$$

$$N_{(\rho)}D_{2(\rho)} = 0. \tag{59}$$

$$\begin{bmatrix} T_{(\rho)} & N_{(\rho)} & F_{(\rho)} & K_{(\rho)} \end{bmatrix} \underbrace{\begin{bmatrix} E & A_{(\rho)} & D_{1(\rho)} & 0 \\ C_{y(\rho)} & 0 & 0 & D_{2(\rho)} \\ 0 & -I & 0 & 0 \\ 0 & -C_{y(\rho)} & -D_{2(\rho)} & 0 \end{bmatrix}}_{\theta_{UI(\rho)}} = \underbrace{\begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}}_{\psi_{UI}}, \tag{60}$$

If $\operatorname{rank} \begin{bmatrix} \theta_{UI(\rho)} \\ \psi_{UI} \end{bmatrix} = \operatorname{rank}(\theta_{UI(\rho)}) \Leftrightarrow \operatorname{rank}(\theta_{UI(\rho)}) = 2n_x + 2n_w$ then

$$\begin{bmatrix} T_{(\rho)} & N_{(\rho)} & F_{(\rho)} & K_{(\rho)} \end{bmatrix} = \psi_{UI}\theta^{\dagger}_{UI(\rho)} - Z_{(\rho)}(I - \theta_{(\rho)}\theta^{\dagger}_{UI(\rho)}) = \Gamma_{1UI(\rho)} - Z_{(\rho)}\Gamma_{2UI(\rho)} \tag{61}$$

## Error Dynamics

$$\begin{cases} \dot{e} & = F_{(\rho)}e + T_{(\rho)}B_{\phi(\rho)}\tilde{\phi} \\ e_z & = C_z e \end{cases} \tag{62}$$

where

$$F_{(\rho)} = \Gamma_{1UI(\rho)}\phi_{AC(\rho)} - Z_{(\rho)}\Gamma_{2UI(\rho)}\phi_{AC(\rho))} \tag{63}$$

$$T_{(\rho)}B_{\phi(\rho)} = \Gamma_{1UI(\rho)}\delta_{\phi(\rho)} - Z_{(\rho)}\Gamma_{2UI(\rho)}\delta_{\phi(\rho)} \tag{64}$$

**Existing conditions**

- R-Detectability

$$\text{rank}\begin{bmatrix} sI - \Gamma_{1UI(\rho)}\phi_{AC(\rho)} \\ \Gamma_{2UI(\rho)}\phi_{AC(\rho)} \end{bmatrix} = n_x \forall\rho \Longrightarrow \text{rank}\begin{bmatrix} sE - A_{(\rho^j)} & D_{1(\rho^j)} \\ C_{(\rho^j)} & D_{2(\rho^j)} \end{bmatrix} = n_x + n_w, \quad \forall j = 1:N, Re(s) > 0. \tag{65}$$

- Existence of General solution:

$$\text{rank}(\theta_{UI(\rho)}) = 2n_x + 2n_w \forall\rho \Longrightarrow \text{rank}\begin{bmatrix} E & D_{1(\rho^j)} & 0 \\ C_{(\rho^j)} & 0 & D_{2(\rho^j)} \\ 0 & -D_{2(\rho^j)} & 0 \end{bmatrix} = n_x + 2n_w, \quad \forall j = 1:N. \tag{66}$$

## Corollary 1.

Under the assumptions (A.1)-(A.3), the design objectives (O.1)-(O.2) are achieved if there exist symmetric positive definite matrices $P_{(\rho)}$ and matrix $Y_{(\rho)}$, positive scalar $\epsilon$ which minimize $\gamma_\infty$ and satisfy that:

$$\begin{bmatrix} \Omega_{11(\rho)} + \eta & \Omega_{12(\rho)} & C_z^T \\ (*) & -\epsilon I & 0 \\ (*) & (*) & -I \end{bmatrix} < 0, \tag{67}$$

then the matrix $Z_{(\rho)}$ is calculated by: $Z_{(\rho)} = -P_{(\rho)}^{-1} Y_{(\rho)}$.

$$\Omega_{11(\rho)} = \sum_i^p \pm\vartheta_i \frac{\partial P_{(\rho)}}{\partial \rho_i} + \mathcal{H}\{P_{(\rho)}\Gamma_{1UI(\rho)}\delta_F + Y_{(\rho)}\Gamma_{2UI(\rho)}\delta_F\}, \tag{68}$$

$$\Omega_{12(\rho)} = P_{(\rho)}\Gamma_{1UI(\rho)}\delta_\phi + Y_{(\rho)}\Gamma_{2UI(\rho)}\delta_\phi, \tag{69}$$

$$\eta = \epsilon(\gamma I)^T(\gamma I), \tag{70}$$

Proof is easily derived from Theorem 1.

# Contents

# Conclusion & Future work

## Conclusion

- A new class of singular NLPV system with Lipschitz nonlinearity is introduced, which promotes the implementation of the LPV framework in modeling the nonlinear system.
- $\mathcal{H}_\infty$ NLPV observer design with parameter-dependent stability is considered to attenuate the disturbance impact on estimation error.

  **Advantages:**
  - Disturbance-decoupling condition is relaxed.
  - Output $y$ can be dependent on $\rho$.
  - $P_{(\rho)}$ in Lyapunov function widens the feasible region of LMI solution.

  **Drawbacks:**
  - $\mathcal{H}_\infty$ performance is not always good if optimized vectors has high dimension or output disturbance exists.
  - $\gamma$ in Lipschitz constraint is maximal boundness, which can make solution conservative.
  - Assumption (A.1) for the boundness of $\dot{\rho}$ can not be always satisfied. For example, $\rho = u$, control input which varies so fast due to controller/environment.
  - Assimption (A.3) for condition is also restrictive comparing to impulse observability.
- Numerical simulation has proven the capability of the proposed observer design in attenuating the disturbance impact under the existence of Lipschitz nonlinearity

# Future work

- Existence of uncertainty to stability of S-NLPV
  - Parametric uncertainty:

$$\begin{cases} E\dot{x} & = (A_{(\rho)} + \Delta A_{(\rho)})x + B_{(\rho)}u + B_{\phi(\rho)}\phi(x,u) + D_{1(\rho)}w \\ y & = C_{y(\rho)}x + D_{2(\rho)}w \end{cases} \tag{71}$$

where $\Delta A_{(\rho)}$ is uncertain term.
  - Uncertainty of $\rho$, i.e. incorrect estimation/measurement $\rightarrow \hat{\rho}$ is input for observer:

$$\begin{cases} \dot{\xi} & = F_{(\hat{\rho})}\xi + J_{(\hat{\rho})}u + L_{(\rho)}y + T_{(\hat{\rho})}B_{\phi(\hat{\rho})}\phi(\hat{x},u) \\ \hat{x} & = \xi + N_{(\hat{\rho})}y \\ \hat{z} & = C_z\hat{x} \end{cases}, \tag{72}$$

- Impact of the time-delay problem on S-NLPV

$$\begin{cases} E\dot{x}(t) & = A_{(\rho)}x(t) + A_{d(\rho)}x(t-h(t)) + B_{(\rho)}u(t) + B_{d(\rho)}u(t-h(t)) \\ & \quad +\Phi(x(t), x(t-h(t)), u(t), u(t-h(t)), t, h(t), \rho(t)) + D_{1(\rho)}w(t) \\ y(t) & = Cx(t) + D_2w(t) \\ x(t) & = \varpi_x(t), t \in [-\bar{h}, 0] \end{cases}$$

$\Longrightarrow$ Submission for IEEE-TAC.

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# $\mathcal{H}_\infty$ observer design for Singular Nonlinear Parameter-varying System

Journée du Groupe de Travail "Sûreté - Surveillance - Supervision"

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# Proof for Theorem 1

Choose the parameter-dependent LPV Lyapunov functional Apkarian et al. [1995]:

$$V_{(\rho)} = e^T P_{(\rho)} e \tag{73}$$

where $P_{(\rho)} > 0$ and $\dot{V}_{(\rho)} = e^T \frac{\partial P_{(\rho)}}{\partial t} e + e^T P_{(\rho)} \dot{e} + \dot{e}^T P_{(\rho)} e$.
Combined with the above Lyapunov function, the sufficient condition for disturbance attenuation (28) can be rewritten as (Wu et al. [1996]):

$$J_\infty = \dot{V}_{(\rho)} + e_z^T e_z - \gamma_\infty w^{*T} w^* < 0. \tag{74}$$

Also, the Lipschitz condition (4) yields the constraint:

$$\|\tilde{\phi}\| \leq \gamma \|e\| \Rightarrow J = (\tilde{\phi})^T (\tilde{\phi}) - e^T (\gamma I)^T (\gamma I) e \leq 0, \tag{75}$$

By applying the S-procedure (Boyd et al. [1994]), the two above constraints in Eqs. (74)-(75) can be achieved if there exists a positive scalar $\epsilon$ such that:

$$J_\infty - \epsilon J < 0 \tag{76}$$

Using error dynamics (23), we obtain:

$$\dot{V}_{(\rho)} \leq \Upsilon^T \begin{bmatrix} \Omega_{11(\rho)} & \Omega_{12(\rho)} & \begin{bmatrix} \Omega_{13(\rho)} & \Omega_{14(\rho)} \end{bmatrix} \\ (*) & (*) & 0 \\ (*) & (*) & 0 \end{bmatrix} \Upsilon = \Upsilon^T \Omega_{(\rho)} \Upsilon, \tag{77}$$

where $\Upsilon = \begin{bmatrix} e^T & \tilde{\phi}^T & w^{*T} \end{bmatrix}^T$.
The constraint (76) $J_\infty - \epsilon J < 0$ is guaranteed if:

$$\Upsilon^T \Omega_{(\rho)} \Upsilon + e^T C_z^T C_z e - \gamma_\infty w^{*T} w^* - \epsilon(\tilde{\phi})^T(\tilde{\phi}) + \epsilon e^T (\gamma I)^T (\gamma I) e < 0, \tag{78}$$

which is equivalent to the following LMI $\forall \Upsilon \neq 0$:

$$\begin{bmatrix} \Omega'_{11(\rho)} + C_z^T C_z + \eta & \Omega_{12(\rho)} & \Omega_{13(\rho)} & \Omega_{14(\rho)} \\ (*) & -\epsilon I & 0 & 0 \\ (*) & (*) & -\gamma_\infty I & 0 \\ (*) & (*) & (*) & -\gamma_\infty I \end{bmatrix} < 0, \tag{79}$$

where $\Omega'_{11(\rho)} = \dot{\rho}\frac{\partial P_{(\rho)}}{\partial \rho} + \mathcal{H}\{P_{(\rho)}\Gamma_{1(\rho)}\delta_F + Y_{(\rho)}\Gamma_{2(\rho)}\delta_F\}$.
To avoid directly handling the derivative $\dot{\rho}$, as mentioned in Wu et al. [1996]

$$\dot{\rho}\frac{\partial P_{(\rho)}}{\partial \rho} \to \sum_{i}^{p} \pm \vartheta_i \frac{\partial P_{(\rho)}}{\partial \rho_i} \tag{80}$$

Apply the Schur Complement, the simplified condition (29) is verified.
$\Longrightarrow$ Proof is completed.