Set-membership Observation and Zonotopes

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ENSEA

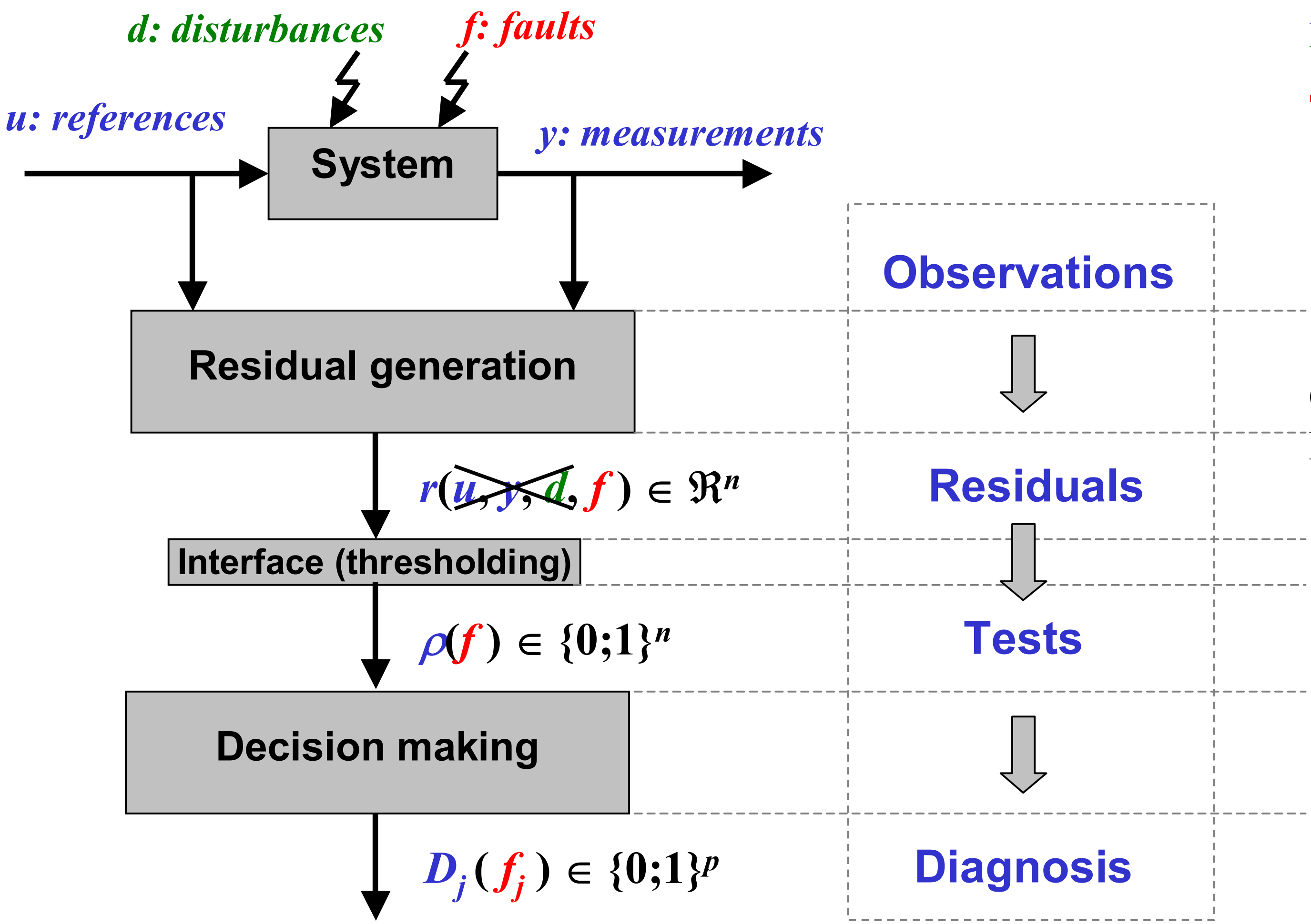
ECS
EQUIPE COMMANDE DES SYSTEMES

Outline

- Introduction to state bounding observation
- Zonotopes
- The linear discrete case (academic examples)
- The non linear continuous case (bioreactor)
- Conclusion & future prospects

Fault diagnosis: The usual « FDI » approach

- **Fault Diagnosis = Fault Detection, Isolation (and Identification)**



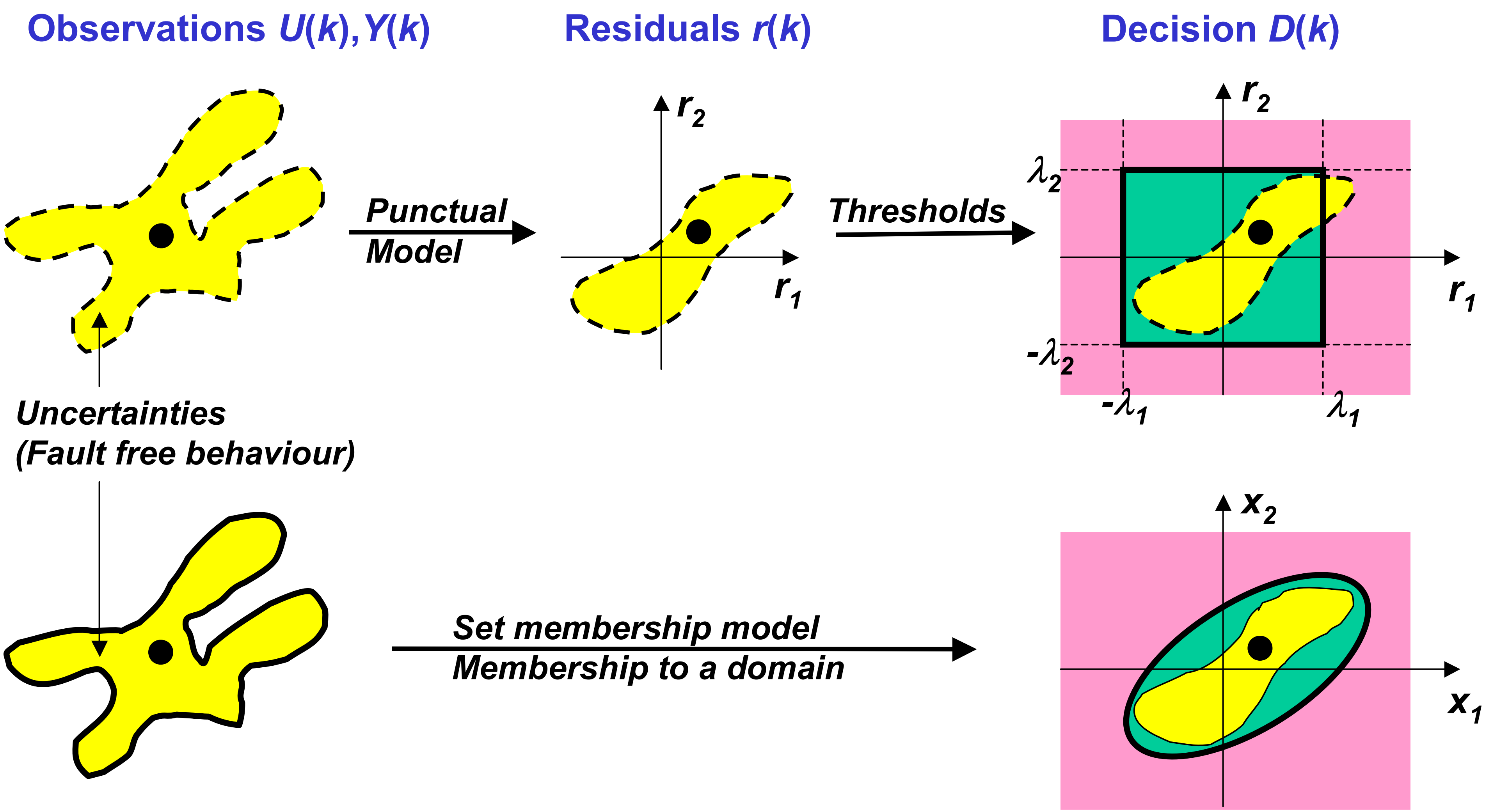
Known value
Unknown value
Tested value
Decoupling

$$\begin{cases} \dot{x} = A.x + B.u + E.d + F.f \\ y = C.x + D.u + G.d + H.f \end{cases}$$

Observers,
Parity relations,…

	f_1	f_2	f_3
ρ_1	1	1	1
ρ_2	1	0	1
ρ_3	1	1	0

Detection



Refined/rough over approximation ➔ High/low sensitivity (guaranteed robustness)

Introduction

Observation:

Data (*corrupted*)	+	**Knowledge** (*partial*)	→	**Information** (*imprecise*)

Measurement (*noise*)	+	**Model** (*modeling errors, disturbances*)	→	**State** (?)

State uncertainty:

- **Not made explicit**
- **Stochastic context (Kalman filter)**
- **Deterministic context** → **Set-membership approaches**

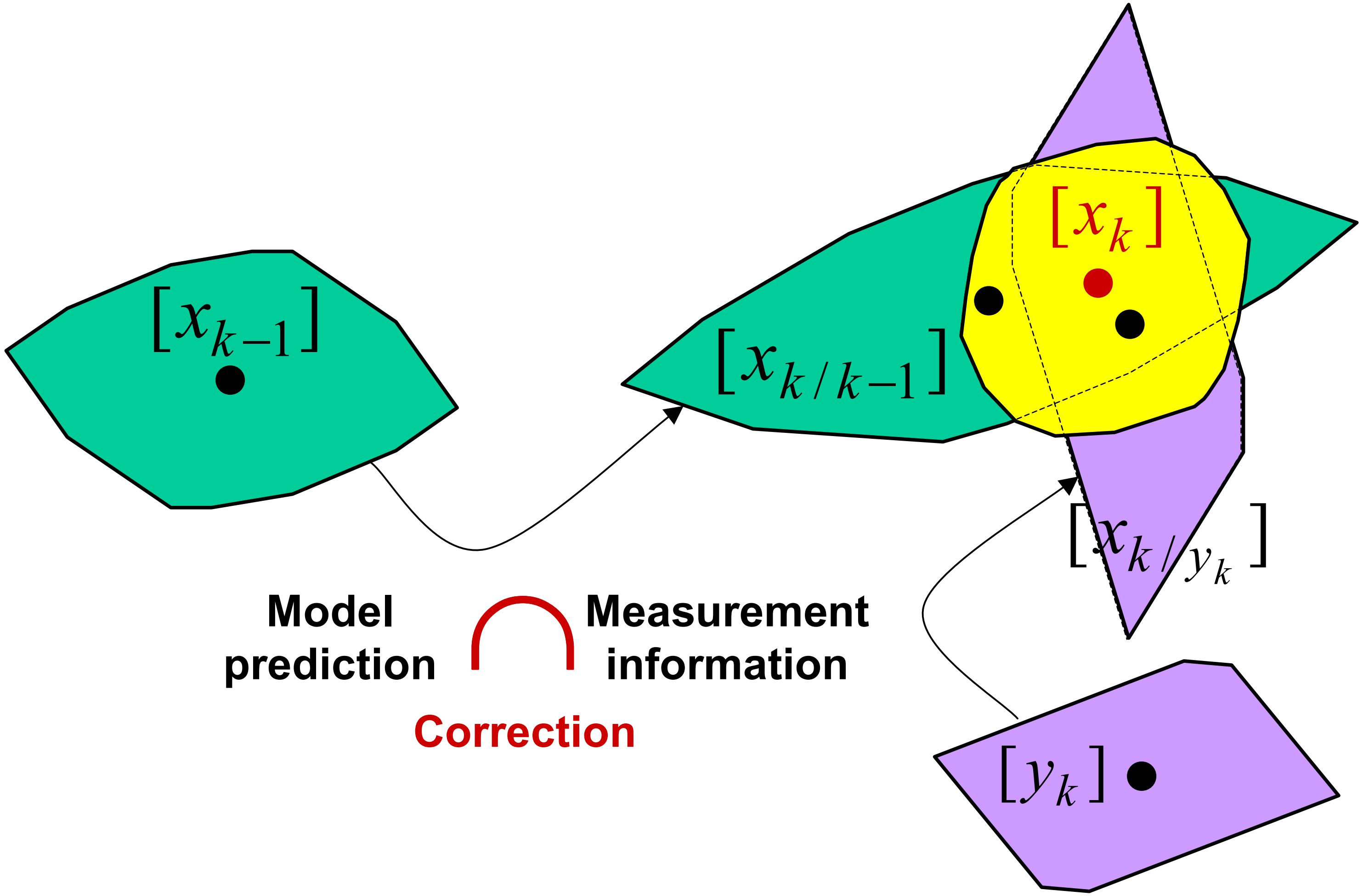
Problem formulation n°1

- **Model of the system (enclosing the real behavior):**

$$\boxed{\begin{aligned} x_{k+1} &= A_k.x_k + B_k.u_k + E_k.v_k \\ y_k &= C_k.x_k + D_k.u_k + F_k.w_k \end{aligned}} \qquad \begin{aligned} v_k &\in \square^n \\ w_k &\in \square^m \end{aligned} \qquad \square = [-1;1]$$

- **Known:** $A_k, B_k, C_k, D_k, E_k, F_k,$ $u_k, y_k,$ $[x_0]$ / $x_0 \in [x_0]$
- **Unknown:** x_k, v_k, w_k
- **Goal:** $[x_k]$, **(smallest) domain such that** $x_k \in [x_k]$ **is guaranteed.**

Bounded error state estimation



Compromise (for guarantee to be achieved):
↑ exactness ⇔ ↑ complexity ⇔ ↓ outer approximation

Main steps of the observation algorithm

Initialization ($[x_0] \leftarrow$ domain s.t. $x_0 \in [x_0]$)

For k=1 to k_{max}

System at time k: u_k, y_k

$[x_{k/k\text{-}1}] \leftarrow$ **Prediction**($[x_{k\text{-}1}]$)

$[x_{k/k\text{-}1}] \leftarrow$ **Reduction**($[x_{k/k\text{-}1}]$)

$[x_k] \leftarrow$ **Correction**($[x_{k/k\text{-}1}]$, y_k, u_k)

End

Domain representation

- **Solutions based on domain approximations:**
  - **Orthotope (i.e. interval vector or boxes)**
  - **Parallelotope** [Chisci, Garulli, Zappa]
  - **Ellipsoïd** [Fogel, Huang, Durieu, Lesecq]
  - **Zonotope** [Kühn] (ODE, wrapping effect)
  - **Polytope (+simplification)** [Walter, Piet-Lahanier]

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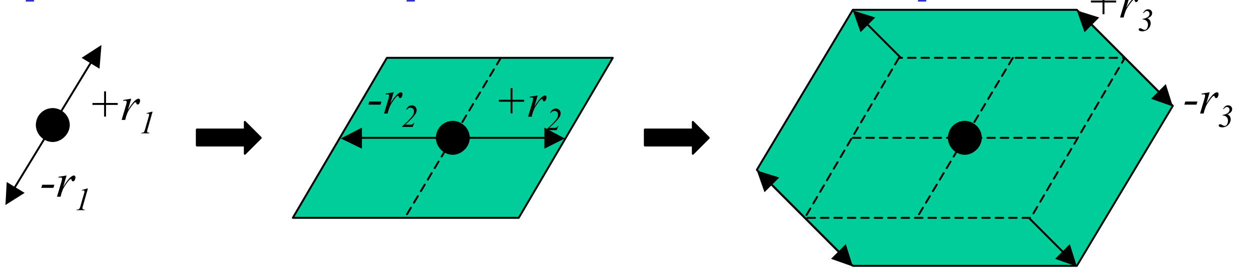
Zonotopes : definitions

- **Zonotope = Minkowski sum of straight line segments**

$$[Z] = [S_1] + \cdots + [S_p]$$

$$[S_i] = c_i + r_i.\square \qquad \square = [-1;+1]$$

- **Exemple (*n*=2): *p*=1 *p*=2 *p*=3**



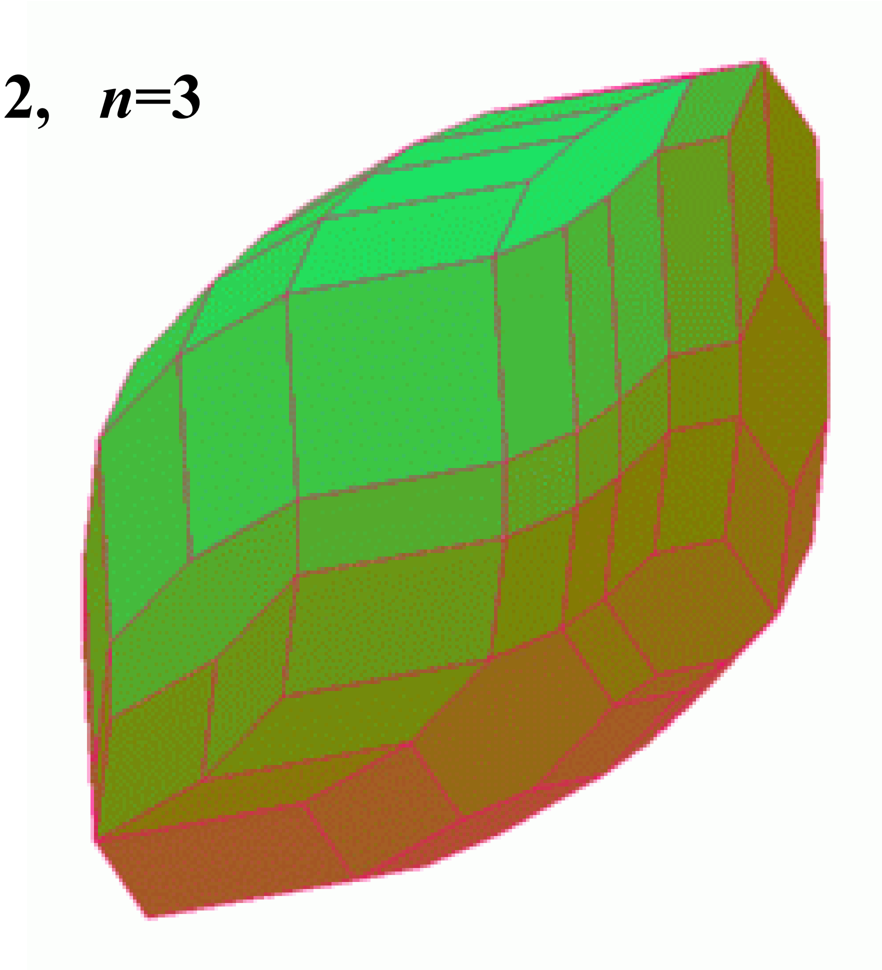
- **Zonotope = Linear image of a *p*-hypercube in a *n*-space (abstract space)**

$$[Z] = c + \underbrace{R.\square^p}_{\blacklozenge R}$$

$$[Z] \subset \Re^n \qquad c \in \Re^n \qquad R \in \Re^{n \times p}$$

Zonotopes : an example

p=12, *n*=3



http://www.decatur.de/wolfgang/zono/index.html

(Centered) Zonotopes : Properties

- **Sum of two zonotopes :**

$$\blacklozenge R_1 + \blacklozenge R_2 = \blacklozenge [R_1\ R_2]$$

➔ **matrix concatenation**

- **Image of a zonotope by a linear application *L* :**

$$L.(\blacklozenge R) = \blacklozenge (L.R)$$

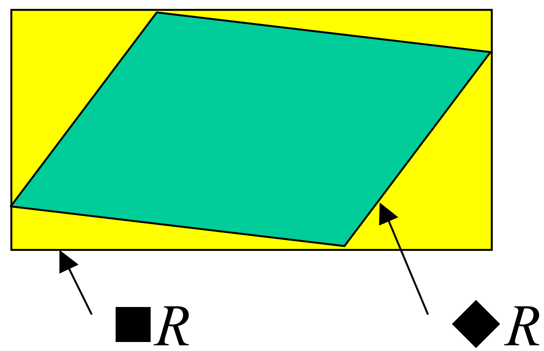
➔ **matrix product**

- **Smallest « box » containing a zonotope (« interval hull »):**

$$\blacksquare R = \blacklozenge rs(R)$$

➔ **Norm of each line vector**

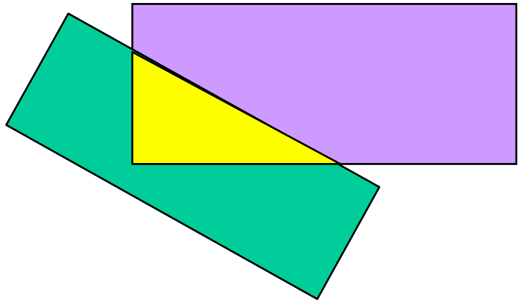
$$rs(R) = \begin{array}{c} \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix} \\ \qquad i \end{array} \qquad rs(R)_{ii} = \sum_{j=1}^{p} \left| R_{ij} \right|$$



Intersection of Zonotopes

- **The intersection of two zonotopes is not always a zonotope:**

➔ **An example:**



- **A solution: outer approx. of the intersection by a zonotope**

$$(c_i + \blacklozenge Z_i) = (c_1 + \blacklozenge Z_1) \,\underline{\cap}\, (c_2 + \blacklozenge Z_2)$$

➔ **Singular Value Decomposition and simple matrix operations.**

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$[x_{k/k\text{-}1}] \leftarrow$ **Reduction**($[x_{k/k\text{-}1}]$)

$[x_k] \leftarrow$ **Correction**($[x_{k/k\text{-}1}]$, y_k, u_k)

End

One step prediction

Recurrence:

- **Hypothesis:** $x_k \in [x_k]$ $\qquad [x_k] = c_k + \blacklozenge R_k$
- **System:** $x_{k+1} = A_k . x_k + B_k . u_k + E_k . v_k \qquad v_k \in \square^n$
- **Therefore:**

$$x_{k+1} \in [x_{k+1/k}] \longrightarrow \left\{ \begin{array}{l} \boxed{c_{k+1/k} = A_k . c_k + B_k . u_k} \\ \boxed{R_{k+1/k} = [A_k R_k \quad E_k]} \end{array} \right.$$

One step prediction

- **Recurrence development (with prediction only):**

$$R_0$$

$$R_1 = \blacklozenge[A_0 R_0 \quad E_0]$$

$$R_2 = \blacklozenge[A_1 A_0 R_0 \quad A_1 E_0 \quad E_1]$$

$$R_3 = \blacklozenge[A_2 A_1 A_0 R_0 \quad A_2 A_1 E_0 \quad A_2 E_1 \quad E_2]$$

etc...

- **Exact solution, but increasing complexity...**
- **Necessity to reduce the domain complexity**

Main steps of the observation algorithm

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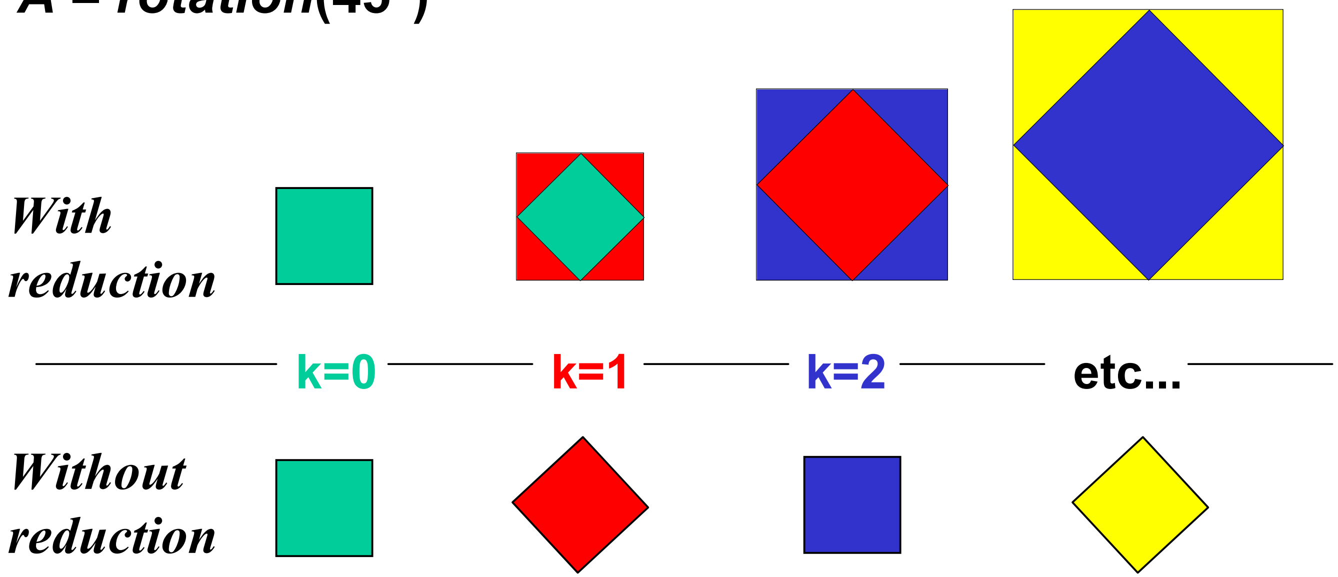
$[x_k] \quad \leftarrow$ **Correction**($[x_{k/k\text{-}1}]$, y_k, u_k)

End

Reduction and wrapping effect

- Reduction ➔ Outer approximation (to keep guarantee)
- Risk of the reduction ➔ wrapping effect
- Example :
  - *reduction*(◆*R*) = ■*R* (smallest «box» containing ◆*R*)
  - *A* = *rotation*(45°)

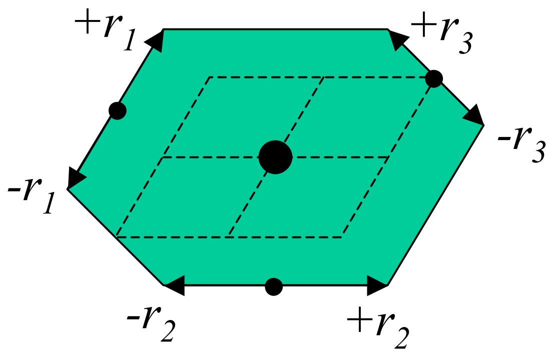


➔ keep as much information on dependencies as possible

Reduction

- **Zonotopes:**

$$\blacklozenge R = \blacklozenge[r_1 \ \cdots \ r_i \cdots \ r_p] \Rightarrow$$



$(p=3)$

- **Choose the zonotope complexity (max. *nd* segments)**

- **Sort on decreasing Euclidian norm:** $\|r_i\| \geq \|r_{i+1}\|$

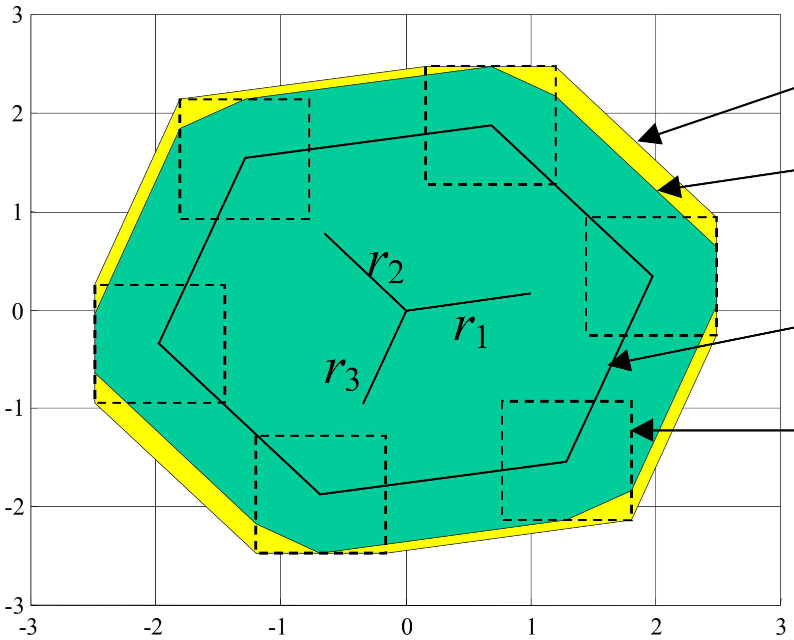
- **Reduction:**

$$Red(\blacklozenge R) = \blacklozenge\left[\boxed{\textit{first (nd-n) col.}}\right] + \blacksquare\left[\boxed{\textit{other col.}}\right]$$

$$Red(\blacklozenge R) = \blacklozenge\left[\boxed{\textit{first (nd-n) col.}}\ \boxed{\textit{n col.}}\right]$$

Reduction : Example

- Reduction of a 6-zonotope into a 5-zonotope:



$Red(\blacklozenge R) = \blacklozenge[r_1, r_2, r_3, rs([r_4, r_5, r_6])]$

$\blacklozenge R = \blacklozenge[r_1, \ldots, r_6]$

$\blacklozenge[r_1 \;\; r_2 \;\; r_3]$ (« Big » segments)

$\blacklozenge N = \blacksquare[r_4, r_5, r_6]$ (« Small » segments)

Main steps of the observation algorithm

Initialization ($[x_0] \leftarrow$ domain s.t. $x_0 \in [x_0]$)

For k=1 to k_{max}

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$[x_{k/k\text{-}1}] \leftarrow$ **Prediction**($[x_{k\text{-}1}]$)

$[x_{k/k\text{-}1}] \leftarrow$ **Reduction**($[x_{k/k\text{-}1}]$)

$[x_k] \leftarrow$ **Correction**($[x_{k/k\text{-}1}]$, y_k, u_k)

End

Correction using measurements

Measurements: Sensors & Domain

$$y(t) = C(t).x(t) + D(t).u(t) + F(t).w(t)$$

$$\Leftrightarrow$$

$$\begin{bmatrix} y_c(t) \\ y_d(t) \end{bmatrix} = \begin{bmatrix} C_c(t) \\ C_d(t) \end{bmatrix} .x(t) + \begin{bmatrix} D_c(t) \\ D_d(t) \end{bmatrix} .u(t) + \begin{bmatrix} F_c(t) & 0 \\ 0 & F_d(t) \end{bmatrix} . \begin{bmatrix} w_c(t) \\ w_d(t) \end{bmatrix}$$

Correction:

$$\boxed{\begin{array}{rl} & (C_{c,k}.c_{k/k-1}) + \blacklozenge(C_{c,k}.Z_{k/k-1})\ldots \\ [x_k] = c_k + \blacklozenge Z_k \leftarrow & \bigcap\ (y_{c,k} - D_{c,k}.u_k) + \blacklozenge(-F_{c,k})\ldots \\ & \bigcap\ (y_{d,k} - D_{d,k}.u_k) + \blacklozenge(-F_{d,k}) \end{array}}$$

← Prediction

← Sensors

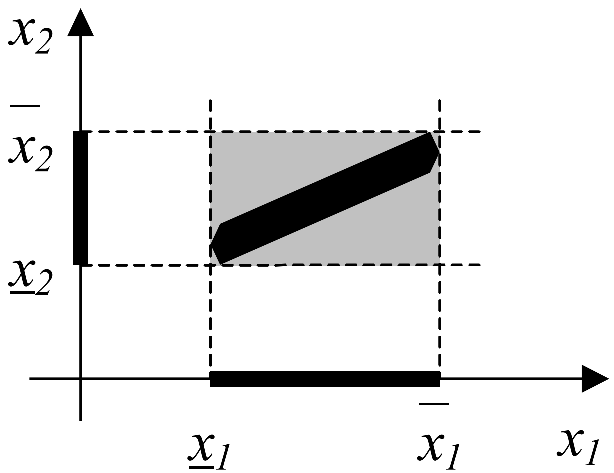
← Domain

Intersection outbounding operator

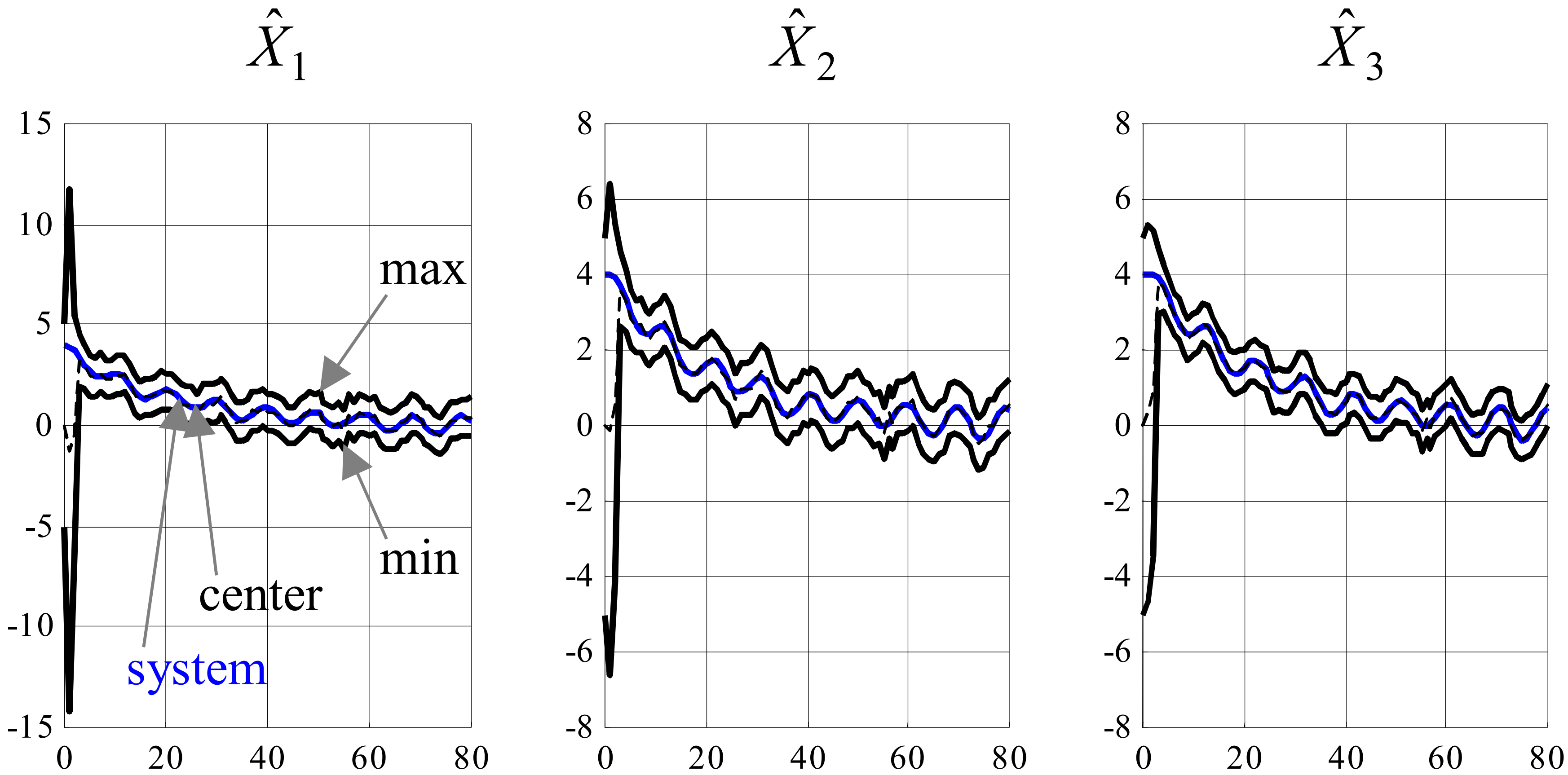
Simulations (academic examples)

- Application to a 3rd order discrete linear model with various observability properties

- Computation of state bounds → Interval hull:

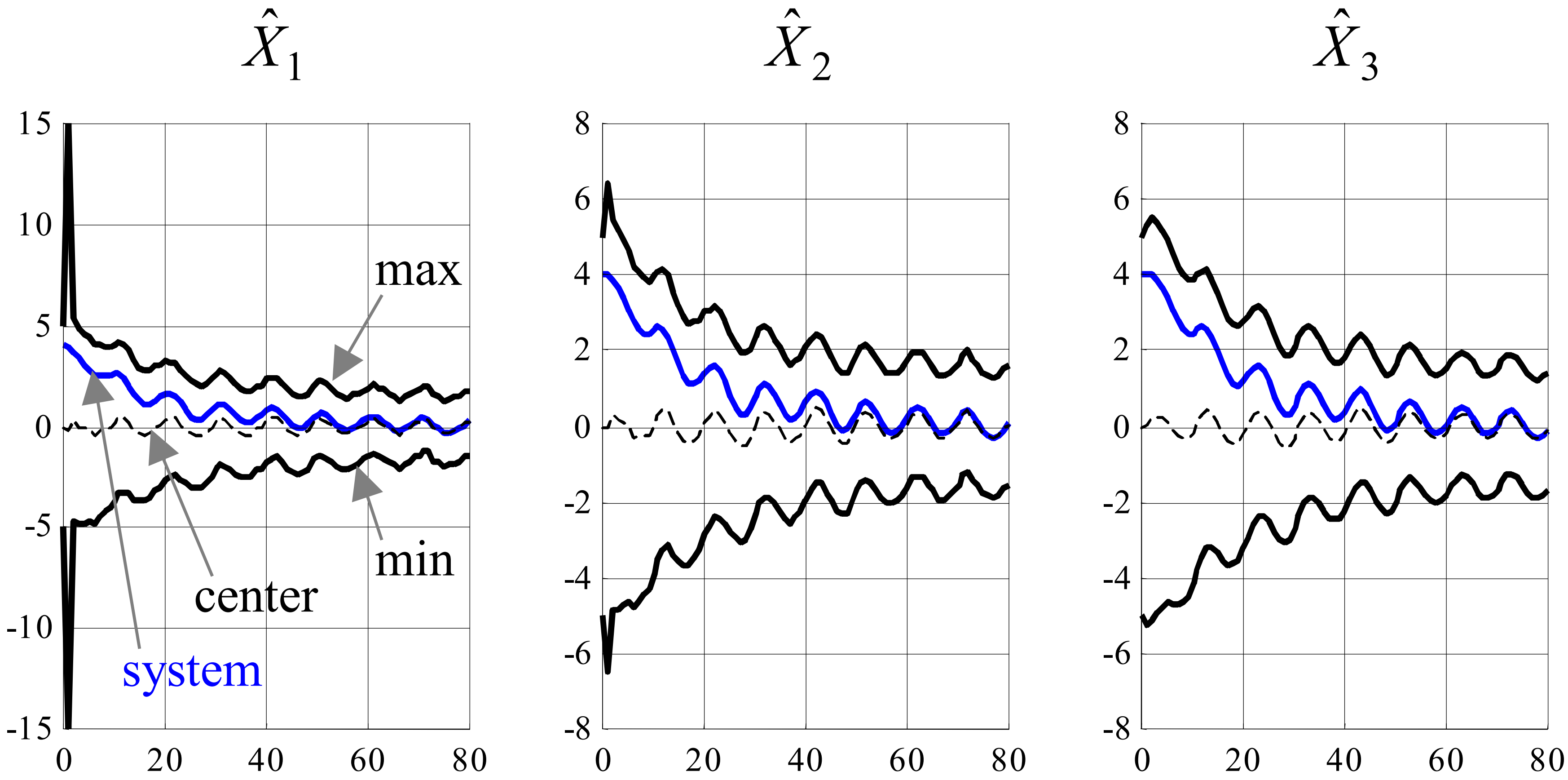


Case 1: Observable



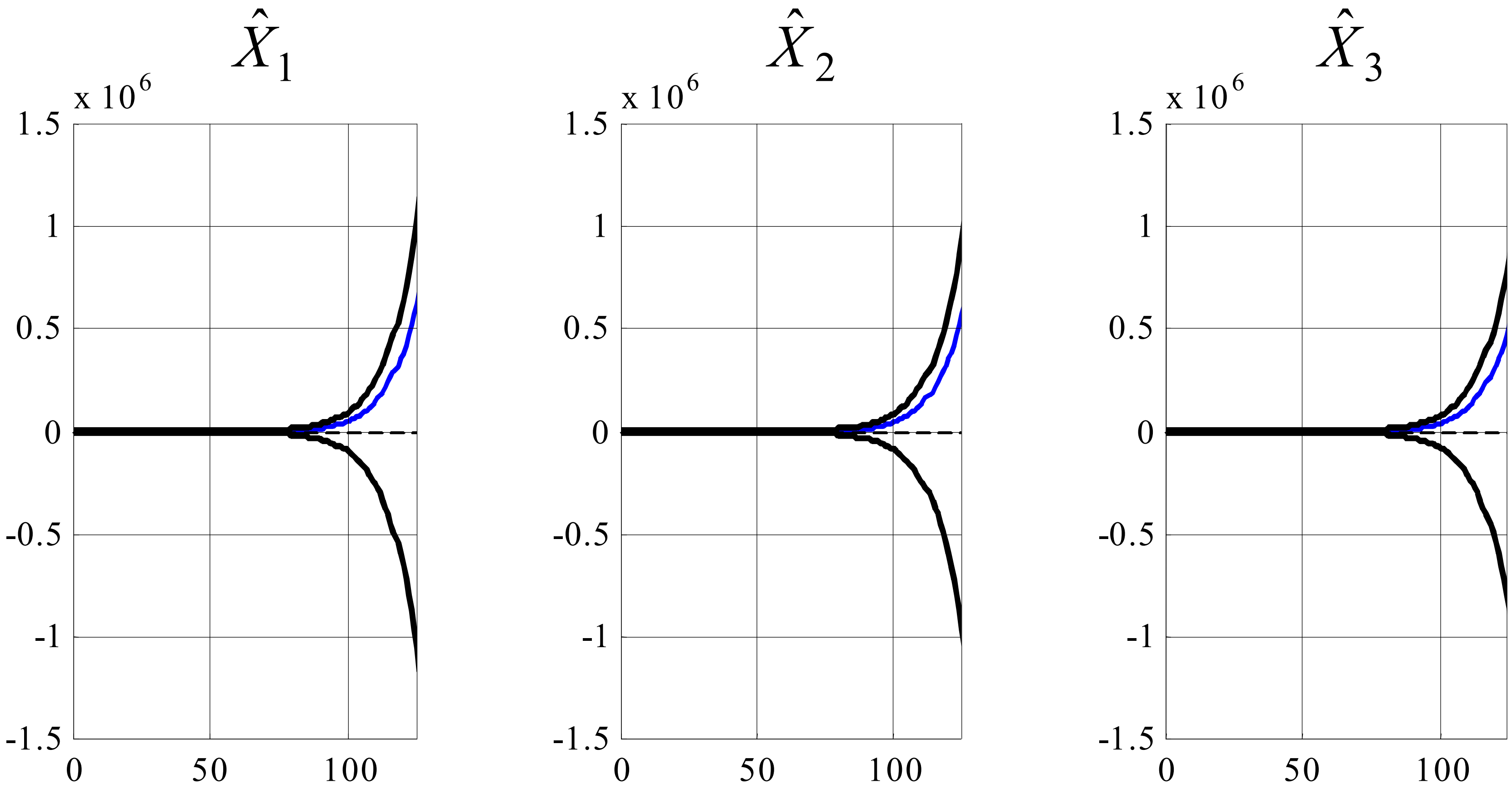
$$G_1(z) = \frac{(z-2)}{(z-0.95)} \cdot \frac{(z-0.7)}{(z-z_0).(z-\bar{z}_0)}$$

Case 2: Non observable, Detectable



$$G_2(z) = \frac{(z-2)}{\cancel{(z-0.95)}} \cdot \frac{\cancel{(z-0.95)}}{(z-z_0).(z-\bar{z}_0)}$$

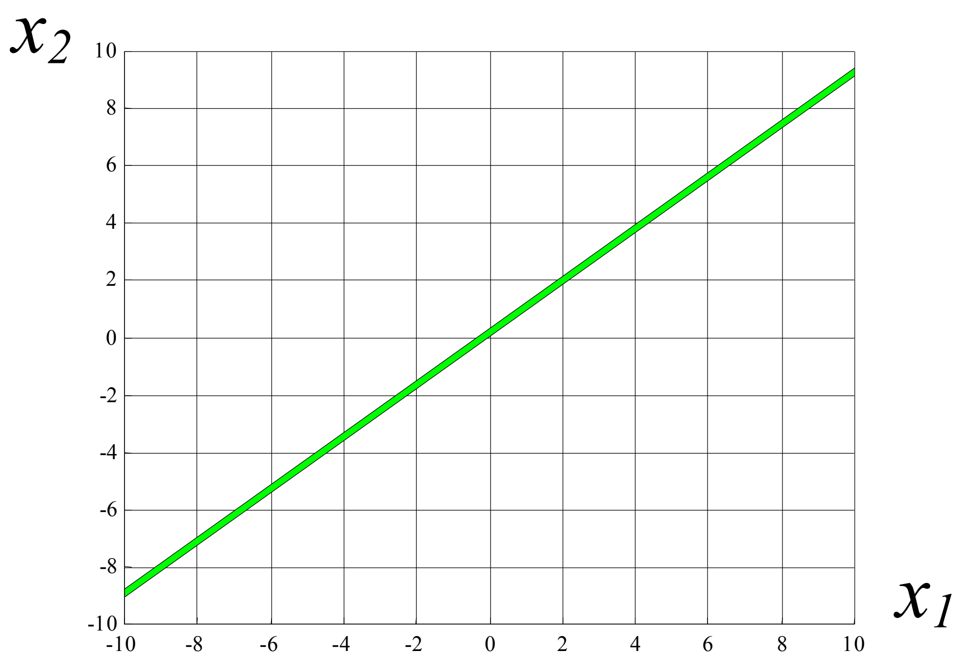
Case 3: Non observable, Non detectable



$$G_3(z) = \frac{(z-2)}{\cancel{(z-1.1)}} \cdot \frac{\cancel{(z-1.1)}}{(z-z_0).(z-\bar{z}_0)}$$

Case 3: Non observable, Non detectable

- Projection of the zonotope in the plane (x_1,x_2):



Zoom, k=30

$$D_{NO}=[0.631\ \ 0.574\ \ 0.522]^T$$

➔ **Domain growth in the direction of the non obs. space**

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Problem formulation n°2

- **Model of the system (enclosing the real behavior):**

$$\boxed{\begin{aligned}\dot{x}(t) &= f(x(t), u(t), v(t)) \\ y(t) &= C.x(t) + D.u(t) + F.w(t)\end{aligned}} \qquad \boxed{\begin{aligned}v(t) &\in [-1;+1]^r \\ w(t) &\in [-1;+1]^m\end{aligned}}$$

- **Sampling:**
  - **Notations:** $x_k = x(kT_s)$, ...
  - **Zero order hold:** $u(t) = u_k, \quad t \in [kT_s\,;\,(k+1)T_s[$

- **Known:** $[x_0]\ /\ x(0) \in [x_0]$ $\quad u_k, y_k$ $\quad \begin{cases} f(.) \in \boldsymbol{C}^2, \text{ local. Lipschitz} \\ C, D, F \quad (F \text{ full line rank}) \end{cases}$

- **Unknown:** $x(t), v(t), w(t)$

- **Goal:** $[x_k]$, **(smallest) domain such that** $x_k \in [x_k]$ **is guaranteed.**

One step prediction

Prediction:

- **From:** $[x_k] = c_k + \blacklozenge R_k$ $\qquad x_k \in [x_k]$
- **And:** $\dot{x}(t) = f(x(t), u(t), v(t))$ $\qquad v(t) \in [-1;+1]^n$
- **Compute:** $[x_{k+1/k}] = c_{k+1/k} + \blacklozenge R_{k+1/k}$ $\qquad x_{k+1} \in [x_{k+1/k}]$

How ?

- **Discretization**
- **Linearization**
- **Inclusion of the linearization error**
- **Inclusion of the discretization error**

Discretization

$$\dot{x}(t) = f(x(t), u(t), v(t))$$

- **Discretization: Euler (for the sake of clarity)**

$$x_{k+1} = x_k + f(x_k, u_k, v_k).T_s + ed_k$$

Discretization error at time *k*

- **Inclusion:**

$$x_{k+1} \in [x_k] + f([x_k], u_k, [v_k]).T_s + [ed_k]$$

$$[x_k] = c_k + \blacklozenge R_k$$
$$[v_k] = [-1;+1]^r$$
$$[ed_k] = cd_k + \blacklozenge Rd_k$$

Inclusion of the discretization error at time *k*

$$x_{k+1} \in \left\{ c_k + R_k.s + f(c_k + R_k.s, u_k, v).T_s, \quad s \in [-1;+1]^p, v \in [-1;+1]^r \right\} + [ed_k]$$

Linearization

$$x_{k+1} \in \left\{ c_k + R_k.s + f(c_k + R_k.s, u_k, v).T_s, \quad s \in [-1;+1]^p, v \in [-1;+1]^r \right\} + [ed_k]$$

Linearization of *f* around (c_k,u_k,0):

$$f(c_k + R_k.s, u_k, v) = f(c_k, u_k, 0) + L_k.R_k.s + M_k.v + el_k(R_k.s, v)$$

$$L_k = \left.\frac{\partial f(x,u,v)}{\partial x}\right|_{\substack{(x,u,v)= \\ (c_k,u_k,0)}} \qquad M_k = \left.\frac{\partial f(x,u,v)}{\partial v}\right|_{\substack{(x,u,v)= \\ (c_k,u_k,0)}}$$

$$[el_k] = cl_k + \blacklozenge Rl_k$$ ← **Inclusion of the linearization error at time *k***

Prediction:

$$\boxed{\begin{aligned} x_{k+1} \in [x_{k+1/k}] &= c_{k+1/k} + \blacklozenge R_{k+1/k} \\ c_{k+1/k} &= c_k + f(c_k, u_k, 0).T_s + cl_k.T_s + cd_k \\ R_{k+1/k} &= [(I + L_k.T_s).R_k \quad M_k.T_s \quad Rl_k.T_s \quad Rd_k] \end{aligned}}$$

Inclusion of the linearization error

- **Linearization error (*):**

$$f(c_k + R_k.s, u_k) = \underbrace{f^{[0]}}_{\substack{\text{Constant} \\ (\text{wrt } s)}} + \underbrace{f^{[1]}(R_k.s)}_{\text{Linear}} + \overbrace{\underbrace{f^{[2]}(R_k.s)}_{\text{Quadratic}} + \underbrace{f_{[3]}(R_k.s)}_{\text{Higher order}}}^{el_k}$$

- **Inclusion:**

$$\boxed{[el_k] = cl_k + \blacklozenge Rl_k = \left(c_k^{[2]} + \blacklozenge R_k^{[2]}\right) + \left(c_{[3]k} + \blacklozenge R_{[3]k}\right)}$$

  - **Quadratic terms :** **disappear but linear dependence between similar quadratic terms is kept.**
  - **Higher order terms : expressed as a non linear function of linear forms + use of interval arithmetic.**

(*) When f depends on v, $c_k \leftarrow [c_k\ 0]^T$, $R_k \leftarrow [R_k\ 0;\ 0\ I]$, $s \leftarrow [s\ v]^T$.

Inclusion of quadratic terms

- **Quadratic form related to the Taylor dev. of f_q around (c_k, u_k):**

$$f_q^{[2]}(c_k, R_k.s, u_k) = (R_k s)^T.Q_q(c_k, u_k).(R_k s)$$

***n*×*n* matrix obtained from a formal calculus software**

$$= s^T.\underline{Q}_{q,k}.s \qquad \underline{Q}_{q,k} = R_k^T.Q_q(c_k, u_k).R_k$$

$$= \sum_i \underline{Q}_{q,k,ii} s_i^2 + \sum_{i<j} (\underline{Q}_{q,k,ij} + \underline{Q}_{q,k,ji}) s_i s_j$$

$$s_i^2 \in [0;+1] = \frac{1}{2} + \frac{[-1;+1]}{2} \qquad s_i s_j \in [-1;+1]$$

Inclusion of quadratic terms

Result:

$$[f^{[2]}(c_k, R_k.s, u_k)] \subseteq c_k^{[2]} + \blacklozenge R_k^{[2]} \qquad c_k^{[2]} = \begin{bmatrix} \vdots \\ c_{q,k}^{[2]} \\ \vdots \end{bmatrix} \quad R_k^{[2]} = \begin{bmatrix} \vdots \\ R_{q,k}^{[2]} \\ \vdots \end{bmatrix}$$

$$c_{q,k}^{[2]} = trace(\underline{Q}_{q,k})/2$$

$$R_{q,k}^{[2]} = [\cdots \quad \underbrace{\underline{Q}_{q,k,ii}/2}_{\forall i} \quad \cdots \mid \cdots \quad \underbrace{(\underline{Q}_{q,k,ij} + \underline{Q}_{q,k,ji})}_{\forall i<j} \quad \cdots]$$

Consequences:

- **Quadratic terms disappear**
- **Linear dependence between similar quadratic terms is kept**

Inclusion of quadratic terms

- **An illustrative example:** (general case in the paper)

$$\forall i, s_i \in [-1;+1]$$

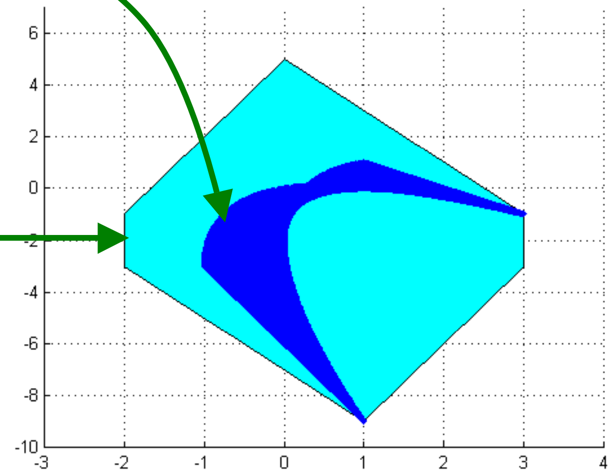
$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} = \begin{bmatrix} -x_1 + (x_1 x_2)/2 \\ -x_2 - x_1 x_2 \end{bmatrix} \Leftarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} \in \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} . + \begin{bmatrix} s_1^2 + s_1 s_2 \\ -2.s_1^2 - 2.s_1 s_2 \end{bmatrix}$$

$$s_1^2 \in [0;+1]$$

$$s_1 s_2 \in [-1;+1]$$

$$\boxed{\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} \in \begin{bmatrix} 1/2 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 1/2 & 1 \\ -3 & -1 & -1 & -2 \end{bmatrix} . \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}}$$



- **Consequences:**
  - Quadratic terms disappear
  - Linear dependence between similar quadratic terms is kept

Inclusion of higher order terms

- Image of a zonotope by a linear form = Interval
- Rest expressed as a non linear function of linear forms:

$$f_{q[3]}(c_k, R_k.s, u_k) = \widetilde{f}_{q[3]}(c_k, \psi(c_k, u_k).R_k.s, u_k)$$

Matrix (formal calculus can be used)
Remark: several solutions for a given application

> **Goal: take linear dependence into account as much as possible before interval inclusion**

$$f_{q[3]}(c_k, R_k.s, u_k) \in \widetilde{f}_{q[3]}(c_k, \blacksquare(\psi(c_k, u_k).R_k), u_k)$$

- Result (use of interval arithmetic):

$$[f_{[3]}(c_k, R_k.s, u_k)] \subseteq c_{[3]k} + \blacklozenge R_{[3]k}$$

Application

Model of a bio-reactor in fed-batch mode (Dochain, 2001):

$$S \rightarrow X \qquad\qquad F_{out} = 0$$

$$\begin{cases} u = D = F_{in} / V \\ \begin{bmatrix} \dot{X} \\ \dot{S} \\ \dot{V} \\ \dot{M} \\ \dot{K} \end{bmatrix} = \dot{x} = f(x,u) = \begin{bmatrix} (\mu - D).X \\ -\mu.X / Y_{xs} + D.(S_{in} - S) \\ D.V - F_{out} \\ 0 \\ 0 \end{bmatrix} \\ y = S + F.w \qquad w \in [-1;+1] \end{cases}$$

Law of Monod:

$$\mu = \mu(K, M, S) = \frac{M.S}{K+S}$$

State extension: parameter estimation

$[x(0)]$: **Initial state (and parametric) uncertainty**

S (resp. ***X***) : concentration of the substrate (resp. biomass). ***D*** : dilution rate. ***V*** : volume. $\boldsymbol{S_{in}}$: input concentration of the substrate. $\boldsymbol{Y_{xs}}$: efficiency coefficient for the growth of the biomass on the substrate. $\boldsymbol{\mu}$: specific growth rate of the biomass. $\boldsymbol{F_{in}}$, $\boldsymbol{F_{out}}$: input and output flow.

Inclusion of the linearization error

- **Higher order terms:**

$$f_{[3]}(c_k, R_k.s, u_k) = \begin{bmatrix} (\mu X)_{[3]}(c_k, R_k.s) \\ -(\mu X)_{[3]}(c_k, R_k.s)/Y_{xs} \\ 0_{3\times 1} \end{bmatrix}$$

$$\boxed{\mu X(x) = \frac{M.S.X}{K+S}}$$

- **Inclusion of $\mu X_{[3]}$:** $[\mu X_{[3]}(c_k, R_k.s)] = c_{I,k} + \blacklozenge R_{I,k}$

- **Result:** $[f_{[3]}(c_k, R_k.s, u_k)] \subseteq c_{[3]k} + \blacklozenge R_{[3]k}$

$$c_{[3]k} = \begin{bmatrix} c_{I,k} \\ -c_{I,k}/Y_{xs} \\ 0_{3\times 1} \end{bmatrix} \qquad R_{[3]k} = \begin{bmatrix} R_{I,k} \\ -R_{I,k}/Y_{xs} \\ 0_{3\times 1} \end{bmatrix}$$

Inclusion of the linearization error

- **Inclusion of $\mu X_{[3]}$:**

$$\mu X(x+\delta x)=\frac{(M+\delta M).(S+\delta S).(X+\delta X)}{(K+\delta K)+(S+\delta S)}\qquad x=\begin{bmatrix}X\\S\\V\\M\\K\end{bmatrix}$$

- **Expression of $\mu X_{[3]}$ as a non linear function of linear forms:**

$$\mu X_{[3]}(x,\delta x)=\frac{(\Psi_1(x).\delta x).(\Psi_2(x).\delta x).(\Psi_3(x).\delta x)}{(\Psi_4(x).x)^3.(\Psi_4(x).(x+\delta x))}$$

Linear forms:

$$\Psi_1(x)=\begin{bmatrix}0 & K & 0 & 0 & -S\end{bmatrix}$$
$$\Psi_2(x)=\begin{bmatrix}(K+S) & -X & 0 & 0 & -X\end{bmatrix}$$
$$\Psi_3(x)=\begin{bmatrix}0 & -M & 0 & (K+S) & -M\end{bmatrix}$$
$$\Psi_4(x)=\Psi_4=\begin{bmatrix}0 & 1 & 0 & 0 & 1\end{bmatrix}$$

- **Interval inclusion:**

$$\begin{cases}x=c_k\\ \delta x=R_k.s\end{cases}\quad\Rightarrow\quad I_{i,k}=\blacksquare(\Psi_i(c_k).R_k)$$

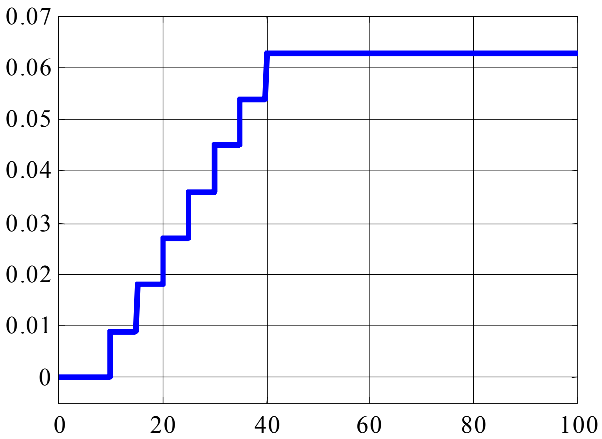
$$[\mu X_{[3]}(c_k,R_k.s)]=\frac{I_{1,k}.I_{2,k}.I_{3,k}}{(\Psi_4.c_k)^3.(\Psi_4.c_k+I_{4,k})}$$

Inclusion of the discretization error

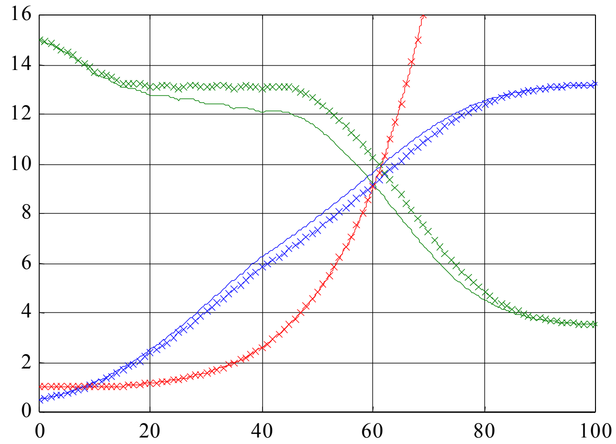
- **Euler scheme:** $ed_k = x_{k+1} - x_k - f(x_k, u_k, v_k).T_s$

- **Empirical evaluation (simulation):**

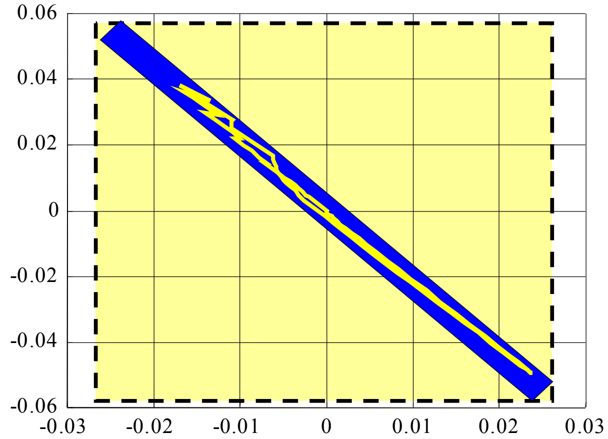
$u(t) = D(t)$



$X(t)$, X_k, $S(t)$, S_k, $V(t)$, V_k



(edX_k, edS_k)



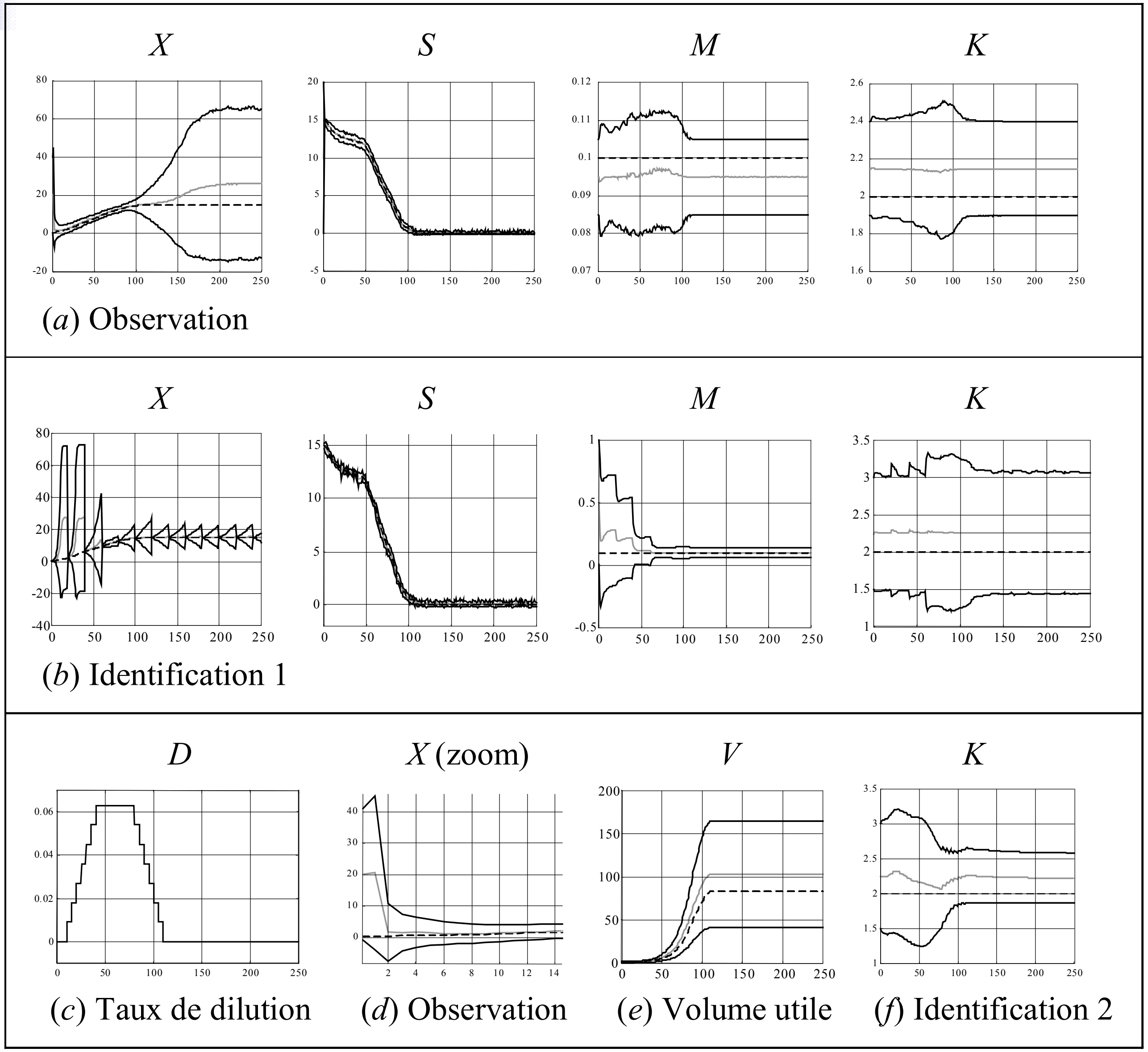
- **Dependency is taken into account, but not the dynamic evol.**

$$[ed_k] = cd_k + \blacklozenge Rd_k \qquad cd_k = 0 \qquad Rd_k = \begin{bmatrix} 0.025 & 0.00125 & 0_{1\times 3} \\ -0.055 & 0.003 & 0_{1\times 3} \\ 0_{3\times 1} & 0_{3\times 1} & 10^{-5}.I_{3\times 3} \end{bmatrix}$$

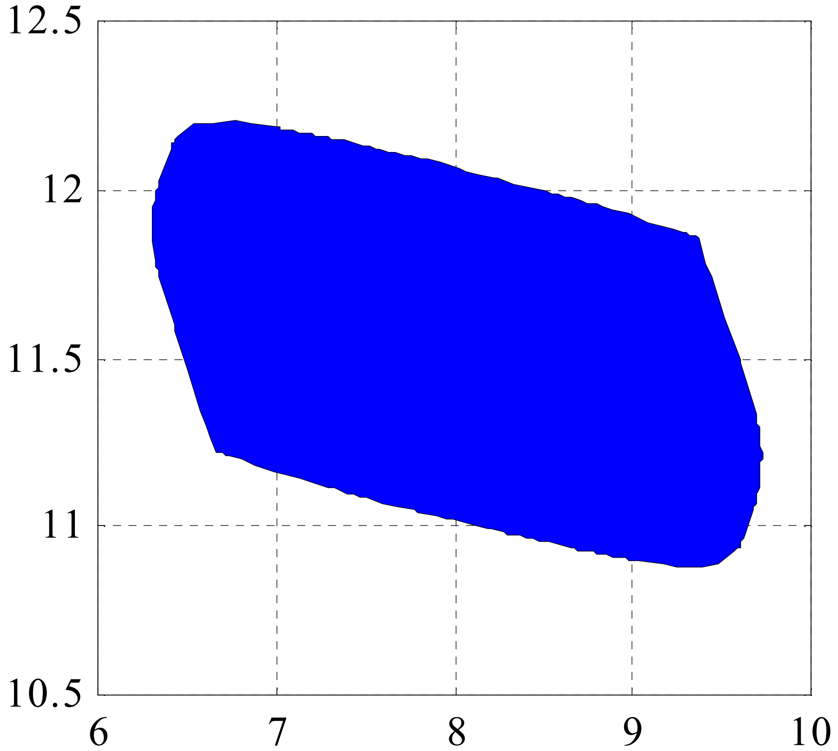
Simulation results: context

	Observation	**Identification 1**	**Identification 2**
Domaine initial $[x_0]$			
X	[0 ; 40]	0.53 ± 0.05	0.5 ± 0.001
S	[0 ; 20]	14.9 ± 0.25	15 ± 0.001
V	[0.5 ; 2]	[0.5 ; 2]	[0.5 ; 2]
M	[0.085 ; 0.105]	[0.01 ; 1]	0.1 ± 0.001
K	[1.9 ; 2.4]	[1.5 ; 3]	[1.5 ; 3]
Instrumentation :			
X	non mesuré	± 0.05 (T_p=20s) (*)	± 0.01 (T_e = 1s)
S	± 0.25 (T_e=1s)	± 0.25 (T_e = 1s)	± 0.01 (T_e = 1s)

Simulation results



A zonotope



Scenario « Observation » : Projection in the (X, S) plane of [x_k] at k = 50

Outline

- Introduction to state bounding observation
- Zonotopes
- The linear discrete case (academic examples)
- The non linear continuous case (bioreactor)
- Conclusion & future prospects

Conclusion & Future prospects

Conclusion:

- Zonotopes ➔ Alternative to other domain representations
- Wrapping effect ➔ Parameterization of the domain complexity
- Observer handling dependency in uncertain non-linear systems with no iterative optimization, no bissection, no facets or vertices enum.
- Need for a refined evaluation of uncertainties ➔ Precision of estimation

Future prospects:

- Dynamic inclusion of the discretization error
- Quantifying the pessimism
- Comparisons with other approaches
- Fault diagnosis

Related publications:

- ECC'2003, NOLCOS'2004, JESA'2005 ? (revised, not accepted at present)