Placement de capteurs pour le recouvrement de l’observabilité des systèmes à commutations

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Outline

1. Introduction to the structural analysis
   - Structural analysis and its graphical approach
   - Structural Controllability: Lin 1974

2. Switched Linear Systems
   - Switched Linear Systems
   - Observability recovering by switching?
   - Matrix decomposition
   - Graphic representation of SSLS

3. Conditions for the existence of sequences guaranteeing observability
   - Connectivity condition
   - Maximal linking condition

4. Observability recovering via sensor placement

5. Conclusions
Structural analysis?

Approaches

\[
\begin{align*}
\text{Algebraic} & \rightarrow \text{rank of pencil matrices} \\
\text{Geometric} & \rightarrow \text{dimension of vectorial subspaces} \\
\text{Structural} & \rightarrow \text{structural properties}
\end{align*}
\]

**Structural Approach**

- Systems in state space or transfer function representation
- Entries of matrices are fixed to zero or free non-zero parameters

**Advantages**

- Systems non specified numerically, in the stage of design or uncertain systems
- Properties are structural and generic

**Covered problems**

- Basic properties of systems (controllability, observability, invariant zeros, subspace dimensions, ...)
- Classical problems of the control theory (input-output decoupling, disturbance rejection, detection and locations of faults, sensor placement, ....)
Structural analysis for linear systems

We study the systems in the state space form:

\[
\Sigma_\Lambda = \begin{cases} 
\dot{x}(t) &= A_\lambda x(t) + B_\lambda u(t) \\
y(t) &= C_\lambda x(t) + D_\lambda u(t) 
\end{cases} 
\]

- \( A_\lambda, B_\lambda, C_\lambda \) and \( D_\lambda \) are real matrices,
- the only knowledge is whether each entry is fixed to zero or an unknown real value represented by a parameter \( \lambda_i \),
- the vector of such parameters is \( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_h\} \) and it can take any value in \( \mathbb{R}^h \),
- a property is true generically for \( (\Sigma_\Lambda) \) if it is true for almost all parameters values \( \lambda_i \).
Graphic representation of a structured linear system

Structured Linear System \( \Sigma_A \) \( \rightarrow \) Directed Graph \( \mathcal{G}(\Sigma_A) = (\mathcal{V}, \mathcal{E}) \)

\( \mathcal{V} = X \cup U \cup Y \) \( \Rightarrow \) Vertex subset

\( \mathcal{E} = A\text{-edges} \cup B\text{-edges} \cup C\text{-edges} \cup D\text{-edges} \) \( \Rightarrow \) Edge subset

for every matrix \( M \) \( \Rightarrow \) \( M\)-edges = \( \{(v_j, v_i) \mid M(i,j) \neq 0\} \) où \( v_j \) is the beginning vertex and \( v_i \) is the end vertex

Example

\[
\begin{align*}
\dot{x}_1 &= \lambda_1 x_2 + \lambda_2 x_4 \\
\dot{x}_2 &= \lambda_3 u_1 \\
\dot{x}_3 &= \lambda_4 x_4 \\
\dot{x}_4 &= \lambda_5 x_2 + \lambda_6 u_2 \\
y_1 &= \lambda_7 x_1 \\
y_2 &= \lambda_8 x_3
\end{align*}
\]
Generic controllability for SISO Systems

Lin identified two structural forms where the system is not generically controllable ([Lin, 1974]):

Form I. $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}; \ b = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}$

Example of Form I:

$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \ b = \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix}$

which corresponds to a graph of type

Form II. $(Ab) = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$

where $P_2$ is $(n - k) \times (n + 1)$ and $P_1$ is $k \times (n + 1)$ with $k \geq 1$ with not more than $k - 1$ non-zero columns.

Example of Form II:

$A = \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{bmatrix}; \ b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

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Example of Form II: \( A = \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{bmatrix} ; b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\)
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\]
which corresponds to a graph of type \( \text{Form I} \).

Form II. \( (Ab) = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \)

where \( P_2 \) is \( (n - k) \times (n + 1) \) and \( P_1 \) is \( k \times (n + 1) \) with \( k \geq 1 \) with not more than \( k - 1 \) non-zero columns.

Example of Form II: \( A = \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{bmatrix} ; b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \)
which corresponds to a graph of type \( \text{Form II} \).
Switched Linear Systems

consist of finite number of continuous-time subsystems and a rule that orchestrates the switching among them [Savkin and Evans, 2002];
Observability recovering by switching: example

Let us consider the SLS with three modes

<table>
<thead>
<tr>
<th>mode 1</th>
<th>mode 2</th>
<th>mode 3</th>
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<tbody>
<tr>
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If the mode sequence is

mode 2  |  mode 1  |  mode 3

(1)
Observability recovering by switching: example

Let us consider the SLS with three modes

\[
\begin{align*}
\dot{x}_1 &= \lambda_1 x_3 + \lambda_2 x_4 + \lambda_3 x_5 \\
\dot{x}_2 &= \lambda_4 x_5 \\
y_1 &= \lambda_6 x_1 + \lambda_7 x_2 \\
y_2 &= 0
\end{align*}
\]

\[
\begin{align*}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= \lambda_5 x_5 \\
y_1 &= \lambda_6 x_1 + \lambda_7 x_2 \\
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\]

If the mode sequence is

\[
\begin{align*}
y_{1,2} &\Rightarrow f_1(x_1, x_2) \\
\dot{y}_{1,2} &\Rightarrow f_2(x_5) \\
y_{2,2} &\Rightarrow f_3(x_3)
\end{align*}
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(1)
Observability recovering by switching: example

Let us consider the SLS with three modes

\[
\begin{array}{ccc}
\text{mode 1} & \text{mode 2} & \text{mode 3} \\
\dot{x}_1 = & \lambda_1 x_3 + \lambda_2 x_4 + \lambda_3 x_5 & \dot{x}_1 = & 0 & \dot{x}_1 = & \lambda_1 x_3 + \lambda_2 x_4 + \lambda_3 x_5 \\
\dot{x}_2 = & \lambda_4 x_5 & \dot{x}_2 = & \lambda_5 x_5 & \dot{x}_2 = & \lambda_5 x_5 \\
y_1 = & \lambda_6 x_1 + \lambda_7 x_2 & y_1 = & \lambda_6 x_1 + \lambda_7 x_2 & y_1 = & \lambda_6 x_1 + \lambda_8 x_2 \\
y_2 = & 0 & y_2 = & \lambda_9 x_3 & y_2 = & \lambda_{10} x_3 \\
\end{array}
\]

If the mode sequence is

\[
\begin{array}{ccc}
\text{mode 2} & \text{mode 1} & \text{mode 3} \\
y_{1,2} \Rightarrow f_1(x_1, x_2) & y_{1,1} \Rightarrow f_1(x_1, x_2) & y_{1,3} \Rightarrow f_1(x_1, x_2) \\
y_{1,2} \Rightarrow f_2(x_5) & y_{1,1} \Rightarrow f_4(x_3, x_4, x_5) & \\
y_{2,2} \Rightarrow f_3(x_3) & y_{2,1} = 0 & \\
\end{array}
\]
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<td>$y_{2,3} = f_7(x_3)$</td>
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the SLS is globally observable.
Structured Switched Linear Systems (SSLS)

\[ \Sigma^\Lambda : \begin{cases} \dot{x}(t) = A(q)x(t), & x(t_0) = x_0 \\ y(t) = C(q)x(t) & \end{cases} \Rightarrow G(\Sigma^\Lambda) = (\mathcal{V}, \mathcal{E}) \] (2)

- A parameter can be common to all modes or to some modes only,
- subset of parameters can be common
Matrix decomposition

\[ A(1) = \begin{bmatrix} 0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 0 & 0 & 0 & \lambda_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A(2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A(3) = \begin{bmatrix} 0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 0 & 0 & 0 & \lambda_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

For the general case, using binary notation, matrix \( A(q) \) can be spread out for mode 1 as:

\[ A(1) = A_{0,0,1} + A_{0,1,1} + A_{1,0,1} + A_{1,1,1} \]

- \( A_{0,0,1} \): specific elements to mode 1
- \( A_{0,1,1} \): common elements between mode 1 and mode 2
- \( A_{1,0,1} \): common elements between mode 1 and mode 3
- \( A_{1,1,1} \): common elements in 3 modes

For the particular case of matrix \( A(1) \) of equation (3),

\[ A(1) = A_{0,0,1} + A_{1,0,1} \]
Matrix decomposition

\[
A(1) = \begin{bmatrix}
0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 \\
0 & 0 & 0 & 0 & \lambda_4 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \quad A(2) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_5 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \quad A(3) = \begin{bmatrix}
0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 \\
0 & 0 & 0 & 0 & \lambda_5 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(3)

For the general case, using binary notation, matrix \(A(q)\) can be spread out for mode 2 as:

\[
A(2) = A_{0,1,0} + A_{0,1,1} + A_{1,1,0} + A_{1,1,1}
\]

specific elements to mode 2 \quad common elements between mode 1 and mode 2

common elements between mode 2 and mode 3 \quad common elements in 3 modes

For the particular case of matrix \(A(2)\) of equation (3),

\[
A(1) = A_{0,0,1} + A_{1,0,1} \quad A(2) = A_{1,1,0}
\]
Matrix decomposition

\[
A(1) = \begin{bmatrix}
0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 \\
0 & 0 & 0 & 0 & \lambda_4 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
A(2) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_5 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
A(3) = \begin{bmatrix}
0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 \\
0 & 0 & 0 & 0 & \lambda_5 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

For the general case, using binary notation, matrix \( A(q) \) can be spread out for mode 3 as:

\[
A(2) = A_{1,0,0} + A_{1,0,1} + A_{1,1,0} + A_{1,1,1}
\]

- Specific elements to mode 3
- Common elements between mode 1 and mode 3
- Common elements between mode 2 and mode 3
- Common elements in 3 modes

For the particular case of matrix \( A(3) \) of equation (3),

\[
A(1) = A_{0,0,1} + A_{1,0,1} \quad A(2) = A_{1,1,0} \quad A(3) = A_{1,0,1} + A_{1,1,0}
\]
Graphic representation of SSLS

\[ A_{1,0,1} = \begin{bmatrix}
 0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 \\
 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 
\end{bmatrix} \]
Graphic representation of SSLS

\[
A_{1,0,1} = \begin{bmatrix}
0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
### Graphic representation of SSLS

The graphic representation of Switched Linear Systems (SSLS) involves understanding the state transitions and observability. A key aspect is the observability matrix, which helps in determining the system's observability properties.

The matrix $A_{1,0,1}$ plays a crucial role in this representation, as it indicates the state transitions and the observability of the system. The matrix is as follows:

$$
\begin{bmatrix}
  x_1 & x_2 & x_3 & x_4 & x_5 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

The highlighted elements such as $1,0,1$ denote the transitions between states and the observability characteristics of the system. This matrix helps in understanding the system's observability properties and the effectiveness of sensor placement.
Graphic representation of SSLS

\[
\begin{align*}
A_{1,0,1} &= \\
\begin{bmatrix}
0 & 0 & w_{1,3}^{1,0,1} & w_{1,4}^{1,0,1} & w_{1,5}^{1,0,1} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{align*}
\]
Graphic representation of SSLS

\[ A_{1,0,1} = \begin{bmatrix} 0 & 0 & w_{1,3}^{1,0,1} & w_{1,4}^{1,0,1} & w_{1,5}^{1,0,1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

Digraph

- \( x_3 \to w_{1,3}^{1,0,1} \)
- \( x_4 \to w_{1,4}^{1,0,1} \)
- \( x_5 \to w_{1,5}^{1,0,1} \)
Graphic representation of SSLS

\[ z_1, z_2, z_3, z_4, z_5 \]

\[ A_{1,0,1} = \begin{bmatrix} 0 & 0 & w_{1,3}^{1,0,1} & w_{1,4}^{1,0,1} & w_{1,5}^{1,0,1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

**Digraph**
Graphic representation of SSLS

\[ A_{1,0,1} = \begin{bmatrix} 0 & 0 & w_{1,3}^{1,0,1} & w_{1,4}^{1,0,1} & w_{1,5}^{1,0,1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

Digraph
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Digraph
Graphic representation of SSLS
Graphic conditions for observability

Necessary and sufficient conditions for the existence of sequences guaranteeing the observability of the SSLS $\Sigma_{SLS}^\Lambda$ are given in [Martinez-Martinez et al., 2011].

$\Rightarrow$ For a SSLS $\Sigma_{SLS}^\Lambda$ with associated graph $G(\Sigma_{SLS}^\Lambda)$ two conditions have to be verified:

- Connectivity and
- Maximal linking
**Connectivity condition**

**Connectivity**

Every $x_i$ is covered by a $Y$-topped path

Some useful definitions

- **Covered vertex** → A vertex which forms part of some path,
- **$Y$-topped path** → A path which final vertex belongs to $Y$
Maximal linking condition

Maximal linking
The maximal number of direct disjoint paths between $X$ and $Y \cup Z$ is equal to $n$.

Some useful definitions

- **Direct path** → A path between vertex subset $X$ and $Y \cup Z$
- **Direct disjoint paths** → Direct paths subset without common vertices
Maximal linking condition

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Some useful definitions

- **Direct path** $\rightarrow$ A path between vertex subset $X$ and $Y \cup Z$
- **Direct disjoint paths** $\rightarrow$ Direct paths subset without common vertices
Observability conditions not verified
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Observability conditions not verified
Solution of the localisation and minimal number of sensor in order to recover de observability based on a completely graphic framework ([Martinez-Martinez et al., 2012])

- Sensor placement to recover the connectivity condition
- Sensor placement to recover maximal linking condition

How many sensors and where?
To recover the output connectivity condition, additional sensors must measure at least one state $x_i$ in any mode for each strongly connected component constituting a minimal unconnected element.

Strongly connected components...

- equivalent elements
- a vertex is equivalent to itself
- ... and minimal unconnected components
- Example
- Strongly connected components
- Partial order: $C_1$ is the infimal unconnected element
Sensor placement for recovering the connectivity condition

To recover the output connectivity condition, additional sensors must measure at least one state $x_i$ in any mode for each **strongly connected component** constituting a **minimal unconnected element**.

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Strongly connected components...

- equivalent elements
- a vertex is equivalent to itself
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Example

- Strongly connected components
- Partial order: $C_1$ is the infimal unconnected element
Guaranteeing connectivity condition
Guaranteeing connectivity condition

\[ x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_5 \rightarrow y_1 \]

\[ w_{1,3}^{1,0,1}, w_{1,4}^{1,0,1}, w_{1,5}^{1,0,1}, w_{2,5}^{0,0,1}, w_{2,5}^{1,1,0} \]

\[ v_1^{1,0,1}, v_2^{0,0,1}, v_2^{1,1,0} \]

\[ z_{1,1}, z_{1,2}, z_{1,3}, z_{2,1}, z_{2,2}, z_{2,3} \]
Sensor placement for recovering the maximal linking condition

**Minimal number of sensors**

To recover the maximal linking condition, the minimal number of additional sensors is equal to $n - \theta(V^+, V^-)$. 

$\theta(V^+, V^-) \rightarrow$ number of edges in maximal subset of direct disjoint paths from $V^+$ to $V^-$
Sensor placement for recovering the maximal linking condition

...but where?

**localisation**

Such additional sensors must measure \( n - \theta(V^+, V^-) \) states in \( V_0^+ \) such that a maximal matching of size \( n \) is obtained in the bipartite graph \( B(\Sigma_{SLS}) \).

Direct graph ⇒ Bipartite graph
Simplifications

\[
\begin{align*}
x_1 & \rightarrow \bullet \\
x_2 & \rightarrow \bullet \\
x_3 & \rightarrow w_1^{1,0,1} \rightarrow \bullet \\
x_4 & \rightarrow w_1^{1,0,1} \rightarrow \bullet \\
x_5 & \rightarrow w_2^{1,1,0} \\
\end{align*}
\]

\[
\begin{align*}
w_1^{1,0,1} & \rightarrow v_1^{1,0,1} \rightarrow \bullet \\
w_1^{1,0,1} & \rightarrow v_1^{1,0,1} \rightarrow \bullet \\
w_1^{1,0,1} & \rightarrow v_1^{1,0,1} \rightarrow \bullet \\
w_2^{0,0,1} & \rightarrow v_2^{0,0,1} \rightarrow \bullet \\
w_2^{1,0,1} & \rightarrow v_2^{1,0,1} \rightarrow \bullet \\
w_2^{1,0,1} & \rightarrow v_2^{1,0,1} \rightarrow \bullet \\
w_2^{1,0,1} & \rightarrow v_2^{1,0,1} \rightarrow \bullet \\
\end{align*}
\]

\[
\begin{align*}
v_1^{1,0,1} & \rightarrow z_{1,1} \\
v_2^{1,0,1} & \rightarrow z_{1,2} \\
v_2^{1,0,1} & \rightarrow z_{1,3} \\
v_2^{1,0,1} & \rightarrow z_{2,1} \\
v_2^{1,0,1} & \rightarrow z_{2,2} \\
v_2^{1,0,1} & \rightarrow z_{2,3} \\
\end{align*}
\]
Simplifications
Bipartite graph

\[
\begin{align*}
\text{Graph 1:} & \\
x_1 \rightarrow v_{1-}^{0,1,0} & \rightarrow y_{1,2} \\
x_2 & \\
x_3 \rightarrow v_{1-}^{1,0,1} & \rightarrow z_{1,1} \\
x_4 & \\
x_5 \rightarrow v_{2-}^{0,0,1} & \rightarrow z_{2,1} \\
\end{align*}
\]

\[
\begin{align*}
\text{Graph 2:} & \\
x_1 \rightarrow v_{1+}^{0,1,0} & \\
x_2 & \\
x_3 \rightarrow v_{1+}^{1,0,1} & \rightarrow z_{1,2} \\
x_4 & \\
x_5 \rightarrow v_{2+}^{0,0,1} & \rightarrow z_{2,2} \\
\end{align*}
\]
Bipartite graph
Guaranteeing maximal linking condition
Guaranteeing maximal linking condition
Guaranteeing maximal linking condition

\[ \begin{align*}
V_0 & \quad \overline{v}_{1,-}^{0,1,0} \quad v_{1,-}^{1,0,1} \\
\overline{x}_1^+ & \quad \overline{x}_2^+ \quad \overline{x}_3^+ \quad \overline{x}_4^+ \\
\overline{y}_{1,-} & \quad \overline{y}_{2,-} \quad \overline{z}_{2,-} \quad \overline{z}_{2,-} \quad \overline{z}_{2,-} \quad \overline{z}_{2,-} \\
\overline{v}_{2,+} & \quad \overline{v}_{2,+} \quad \overline{v}_{2,+} \quad \overline{v}_{2,+} \\
\overline{x}_5^+ & \quad \overline{v}_{1,+}^{1,0,1} \\
\end{align*} \]
Guaranteeing maximal linking condition
## Conclusions

### Structural approach

- Sensor placement to recover the observability has been extended to the SLS case
- The strong relation between the structure and the observability of SLS has been shown

### Graph approach

- A new graph which represents SSLS has been introduced
- Simple algorithms of graph theory are required to solve the problem of sensor placement (low complexity)
  - Strongly connected components
  - Output connected paths
  - Disjoint paths
  - Maximal number of edges/vertices belonging to a particular path
Merci pour votre attention
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