Fault tolerant control

Joint work with M. Fliess and C. Join

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Contents of Presentation

- Introduction
- A simple example
- A not so simple example
- A Separately excited DC Motor
- A water tank example
- Conclusions

Introduction

Fault diagnosis has been the object of numerous investigations from different viewpoints. Roughly speaking, the existing literature classifies into three general approaches:

- Observer based (estimators, filters) : Beard 1971, Jones 1973, Willsky 1976, White and Speyer 1987, Frank 1990, Nikoukhah 1994, Frank and Ding 1997.
- Parity equations based: Gertler and Singer 1990, Magni and Mouyon 1994, Mevdvedev 1995, Höffing and Iserman 1996.
- Identification and Parameter estimation based: Iserman 1984, Han and Frank 1995, Doraiswami and Stevenson 1996, Alcorta-García *et al* 1999.

Introduction

In this talk we shall explore, through simple examples, the possibilities of an algebraic approach to fault detection in uncertain linear, and nonlinear, systems.

- The linear case, which is based on module theory, is treated in the recent joint article by Fliess, Join and Sira-Ramírez, IJC, Vol. 77, pp. 1223-1242, November 2004.
- The nonlinear case, which is based on differential fields, is also treated by the previously mentioned three authors in an upcoming article celebrating Prof. Dr.-Ing. M. Zeitz' 65th Birthday.

A simple example

Consider the first order constant linear system,

$$\tau \dot{y}(t) + y(t) = K(u(t) + \mathbf{w}(t)) + \vartheta(t)$$

where:

- $u(t)$ and $y(t)$ are, respectively, the control and the output variables,
- $\mathbf{w}(t)$ is the actuator fault variable which must be detected and identified,
- The gain K is known,
- the system time constant τ is assumed to be unknown.
- The perturbation input ϑ is constant but unknown.

A simple example

In operational calculus notation the system reads as

$$(\tau s + 1)y = K(u + \mathbf{w}) + \tau y(0) + \frac{\vartheta}{s}$$

We annihilate the influence of the initial conditions, and of the perturbation input, ϑ, via multiplying out by the differential operator: $\left[\frac{d^2}{ds^2}\right] s$. We obtain,

$$\begin{aligned} K\left(2\frac{d\mathbf{w}}{ds} + s\frac{d^2\mathbf{w}}{ds^2}\right) \quad = \quad & s^2\left(\tau\frac{d^2y}{ds^2}\right) + s\left(\frac{d^2y}{ds^2} + 4\tau\frac{dy}{ds} - K\frac{d^2u}{ds^2}\right) \\ & + \left(2\frac{dy}{ds} + 2\tau y - 2K\frac{du}{ds}\right) \end{aligned}$$

A simple example

The right hand side of the previous expression is zero whenever $\mathbf{w}(t)$ is identically zero but it acquires some value when $\mathbf{w}$ ceases to be zero. The time domain double integral of the right hand side expression:

$$\left(\tau\frac{d^2y}{ds^2}\right) + s^{-1}\left(\frac{d^2y}{ds^2} + 4\tau\frac{dy}{ds} - K\frac{d^2u}{ds^2}\right)$$
$$+s^{-2}\left(2\frac{dy}{ds} + 2\tau y - 2K\frac{du}{ds}\right)$$

acts as *fault detection function*, which can be continuously evaluated in the time domain, provided τ is perfectly known.

A simple example

Since the time constant τ is uncertain, a possible solution to the fault detection problem is to obtain a partity equation which is independent of τ. This can be achieved if we solve for τ from the previous frequency domain parity equation to obtain:

$$\tau = \frac{n(s)}{d(s)}$$

$$\begin{aligned} n(s) &= sK\frac{d^2(u+\mathbf{w})}{ds^2} - s\frac{d^2 y}{ds^2} + 2(K\frac{d(u+\mathbf{w})}{ds} - 2\frac{dy}{ds}) \\ d(s) &= s^2\frac{d^2 y}{ds^2} + 4s\frac{dy}{ds} + 2y \end{aligned}$$

and now use the fact that $\frac{d\tau}{ds} = 0$.

A simple example

The equation $\frac{d\tau}{ds} = 0$ yields:

$$
\begin{aligned}
& s\left(K\frac{d^2\mathbf{w}}{ds^2} + 2K\frac{d\mathbf{w}}{ds}\right)\left(s^2\frac{d^3y}{ds^3} + 6s\frac{d^2y}{ds^2} + 6\frac{dy}{ds}\right) \\
& -\left(sK\frac{d^3\mathbf{w}}{ds^3} + 3K\frac{d^2\mathbf{w}}{ds^2}\right)\left(s^2\frac{d^2y}{ds^2} + 4s\frac{dy}{ds} + 2y\right) \\
& = \quad a(s)b(s) - c(s)d(s)
\end{aligned}
$$

$$
\begin{aligned}
a(s) &= s\left(K\frac{d^3u}{ds^3} - \frac{d^3y}{ds^3}\right) + 3\left(K\frac{d^2u}{ds^2} - \frac{d^2y}{ds^2}\right) \\
b(s) &= \left(s^2\frac{d^2y}{ds^2} + 4s\frac{dy}{ds} + 2y\right) \\
c(s) &= \left(s\left(K\frac{d^2u}{ds^2} - \frac{d^2y}{ds^2}\right) + 2\left(K\frac{du}{ds} - \frac{dy}{ds}\right)\right) \\
d(s) &= \left(s^2\frac{d^3y}{ds^3} + 6s\frac{d^2y}{ds^2} + 6\frac{dy}{ds}\right)
\end{aligned}
$$

A simple example

Multiply now both sides of the last equation by s^{-3}. We obtain:

$$
\begin{aligned}
&s^{-3}\left[s\left(K\frac{d^2\mathbf{w}}{ds^2}+2K\frac{d\mathbf{w}}{ds}\right)\left(s^2\frac{d^3y}{ds^3}+6s\frac{d^2y}{ds^2}+6\frac{dy}{ds}\right)\right.\\
&\left.-\left(sK\frac{d^3\mathbf{w}}{ds^3}+3K\frac{d^2\mathbf{w}}{ds^2}\right)\left(s^2\frac{d^2y}{ds^2}+4s\frac{dy}{ds}+2y\right)\right]\\
&\qquad = \quad s^{-1}a(s)s^{-2}b(s)-s^{-1}c(s)s^{-2}d(s)=\mathbf{r}(s)
\end{aligned}
$$

Denote by $\mathbf{r}$ the right hand side of this last expression,

$$
\begin{aligned}
\mathbf{r}(s) \quad = \quad &\left[\left(K\frac{d^3u}{ds^3}-\frac{d^3y}{ds^3}\right)+3s^{-1}\left(K\frac{d^2u}{ds^2}-\frac{d^2y}{ds^2}\right)\right]\times\left(\frac{d^2y}{ds^2}+4s^{-1}\frac{dy}{ds}+2s^{-2}y\right)\\
-&\left[\left(K\frac{d^2u}{ds^2}-\frac{d^2y}{ds^2}\right)+2s^{-1}\left(K\frac{du}{ds}-\frac{dy}{ds}\right)\right]\times\left[\frac{d^3y}{ds^3}+6s^{-1}\frac{d^2y}{ds^2}+6s^{-2}\frac{dy}{ds}\right]
\end{aligned}
$$

A simple example

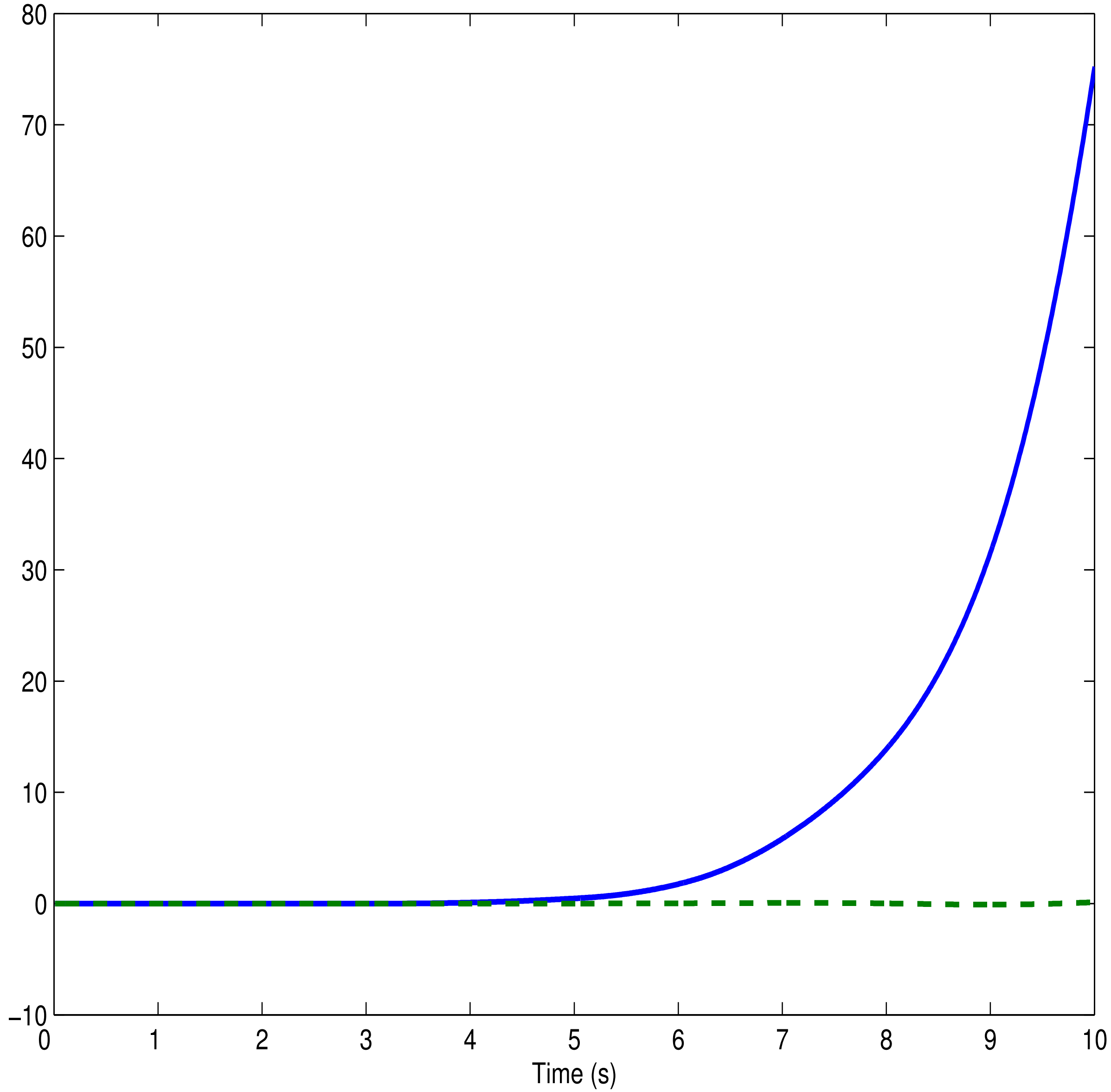
One obtains, in the time domain, the following expression for the residual:

$$
\begin{aligned}
\mathbf{r} \quad = \quad & \left(t^3 y - K t^3 u + 3 \int (K\lambda^2 u - \lambda^2 y)\right) * \left(t^2 y - 4 \int (\lambda y) + 2 \int^{(2)} (y)\right) \\
& - \left(t^2 y - 4 \int (\lambda y) + 2 \int^{(3)} (y)\right) * \left(-t^3 y + 6 \int (t^2 y) - 6 \int^{(2)} (\lambda y)\right)
\end{aligned}
$$

where $\int \square = \int_0^t \square d\lambda$, $\int^{(2)} \square = \int_0^t \int_0^{\lambda_1} \square d\lambda d\lambda_1$, and $*$ stands for the *convolution* operation.

The residual $\mathbf{r}$ is, thus, robust with respect to the uncertain parameter τ.

A simple example



A simple example

A PI controller is clearly accomodating.
We specifically propose a PI with feedforward compensation for the robust tracking the desired trajectory

$$u = u^* + p_1(y^* - y) + \frac{p_2}{s}(y^* - y)$$

with

$$u^* = \frac{1}{K}(\tau s + 1)\, y^*$$

A simple example

In the simulations we have used the following parameters

$$p_1 = 1, \;\; p_2 = 2$$

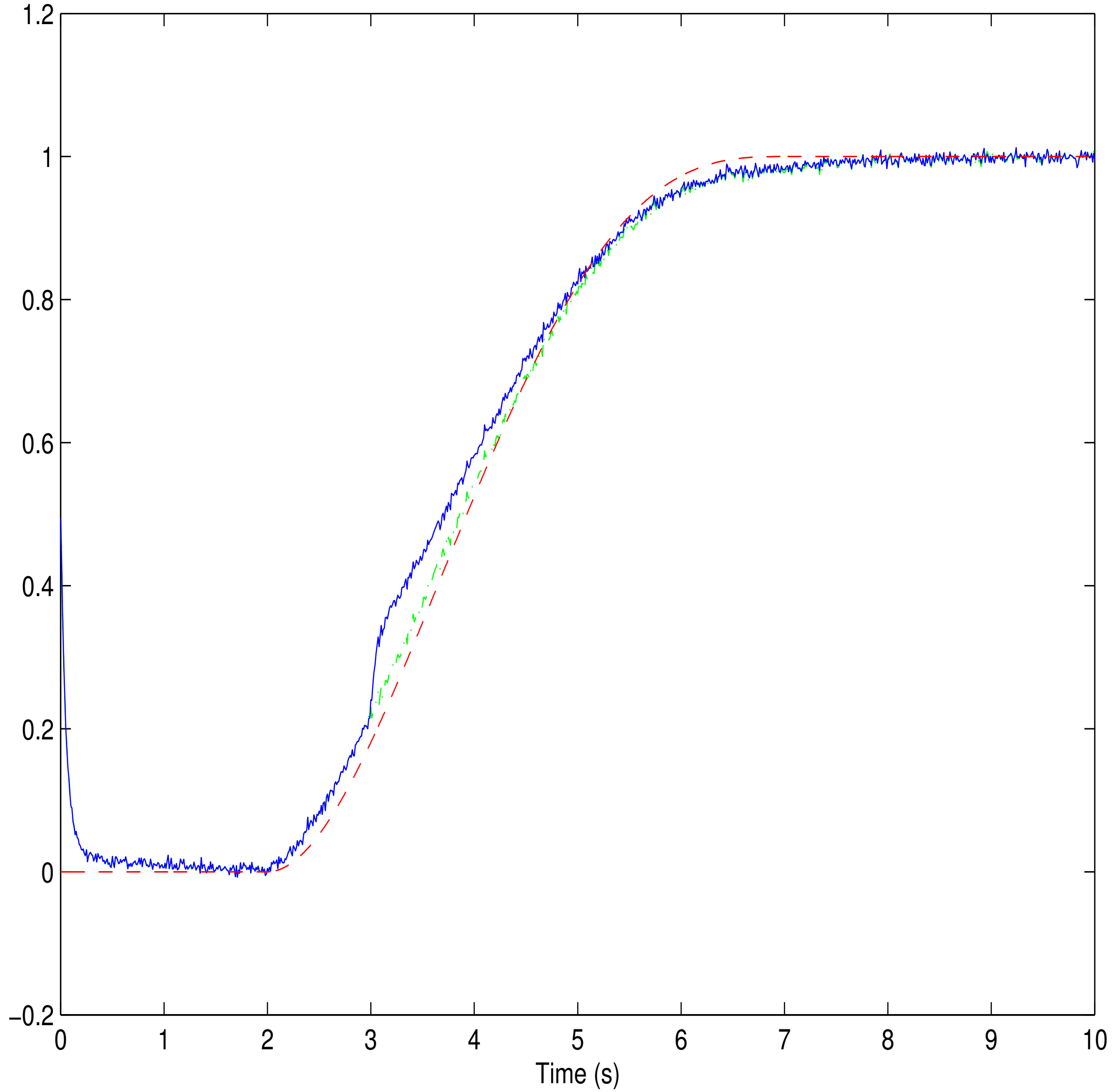
$$y(0) = 0.5, \;\; \varpi = 0.1$$

• The uncertain nominal value of the parameter τ was chosen to be $\tau_{system} = 0.1$, but we used instead $\tau_{model} = 0.5$.

• The actuator fault was taken to be a unit step of amplitude 0.2.

• In the simulations we added a computer generated system noise to the plant dyanmics to see its effects on the proposed scheme.

A simple example



A simple example

The preceeding example, in spite of its simplicity, readily allow us to say

- It is possible to detect faults when the system is operating in closed loop
- It is possible to construct residuals for fault detection which are robust with respect to *parameter uncertainty, initial conditions and constant perturbations*
- We did not use a stochastic formulation of the problem.
- Our emphasis will now be directed towards actually identifying the unknown fault signal $\mathbf{w}(t)$.

Identifying unknown failures

We may propose an output derivative based approach to actually estimate the unknown signal, $\mathbf{w}(t)$, as follows:
Take the time derivative of the system equation in order to annihilate the constant perturbation input ϑ. We obtain:

$$\tau\ddot{y} = -\dot{y} + K(\dot{u} + \dot{\mathbf{w}}(t))$$

Note that

$$\dot{\mathbf{w}}(t) = \frac{1}{K}\left[\tau\ddot{y} + \dot{y} - K\dot{u}\right]$$

and, hence, since $\mathbf{w}(t)$ is zero for some time interval, we set

$$\mathbf{w}_e(t) = \frac{1}{K}\int_0^t \left[\tau\ddot{y}(\sigma) + \dot{y}(\sigma) - K\dot{u}(\sigma)\right] d\sigma, \quad \mathbf{w}_e(0) = 0$$

Identifying unknown failures

We arrive at the same forumla, for the estimate of $\mathbf{w}(t)$, if we use the frequency domain formulation:

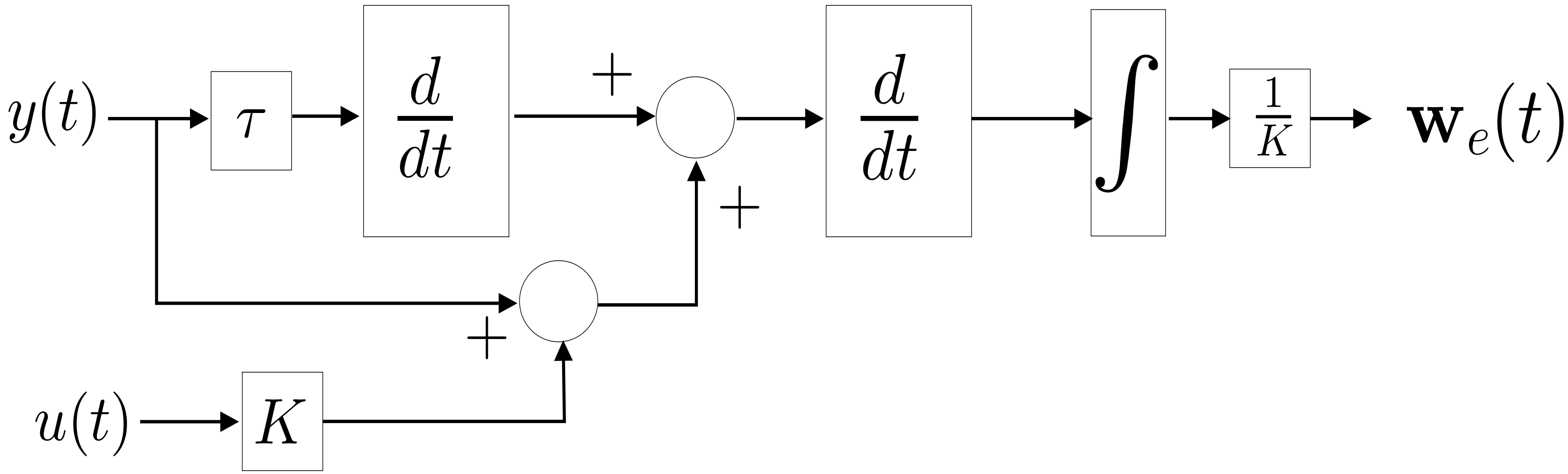
$$(\tau s + 1)y = K(u + \mathbf{w}) + \tau y(0) + \frac{\vartheta}{s}$$

Multiplying out by the differential operator: $\left[\frac{d^2}{ds^2}\right] s$ to obtain, once more,

$$\begin{aligned} K\left(2\frac{d\mathbf{w}}{ds} + s\frac{d^2\mathbf{w}}{ds^2}\right) &= s^2\left(\tau\frac{d^2y}{ds^2}\right) + s\left(\frac{d^2y}{ds^2} + 4\tau\frac{dy}{ds} - K\frac{d^2u}{ds^2}\right) \\ &+ \left(2\frac{dy}{ds} + 2\tau y - 2K\frac{du}{ds}\right) \end{aligned}$$

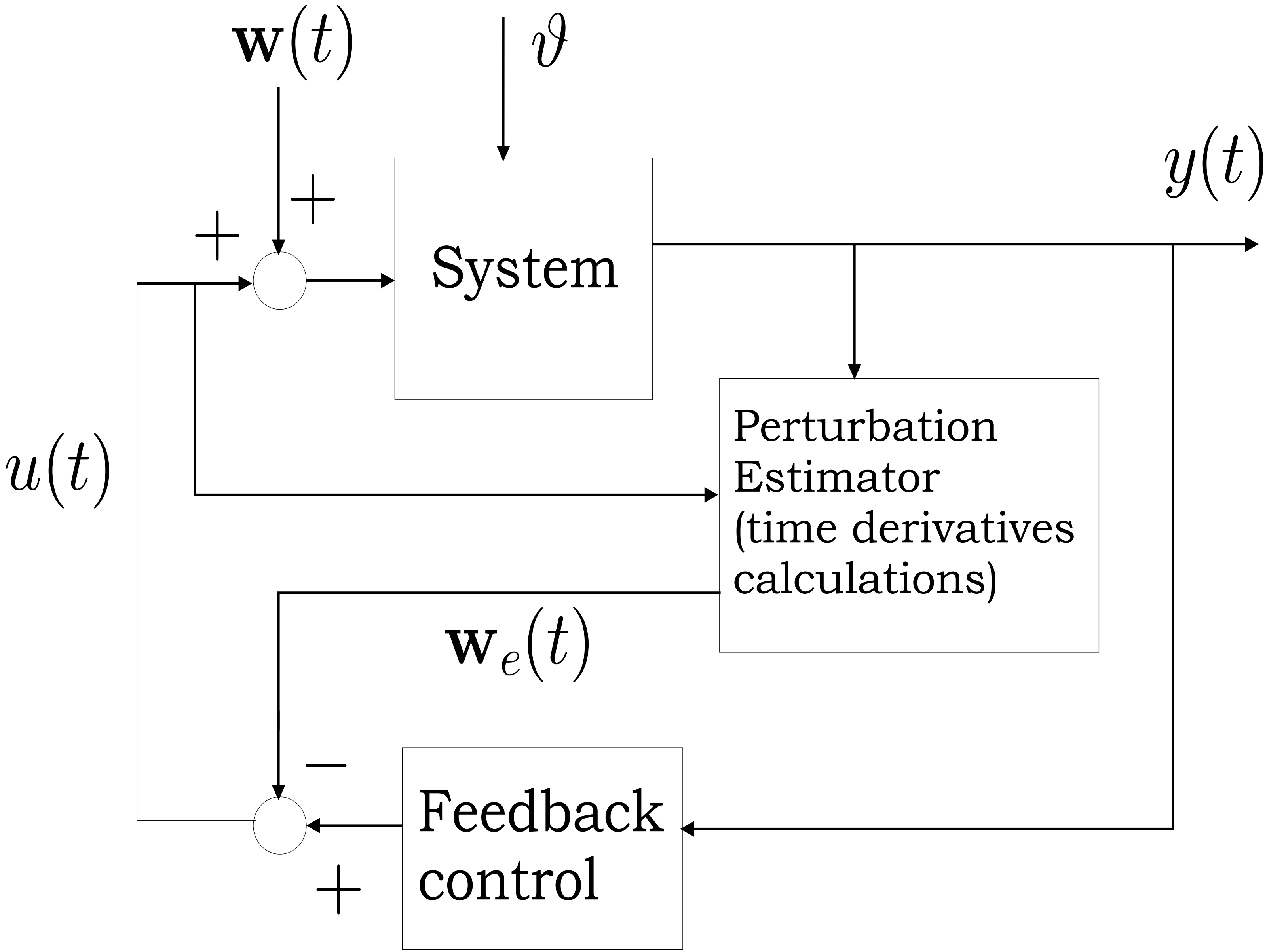
The time domain equivalent of this expression is precisely our previous integration formula.

An application...



Perturbation input signal estimator

An application...



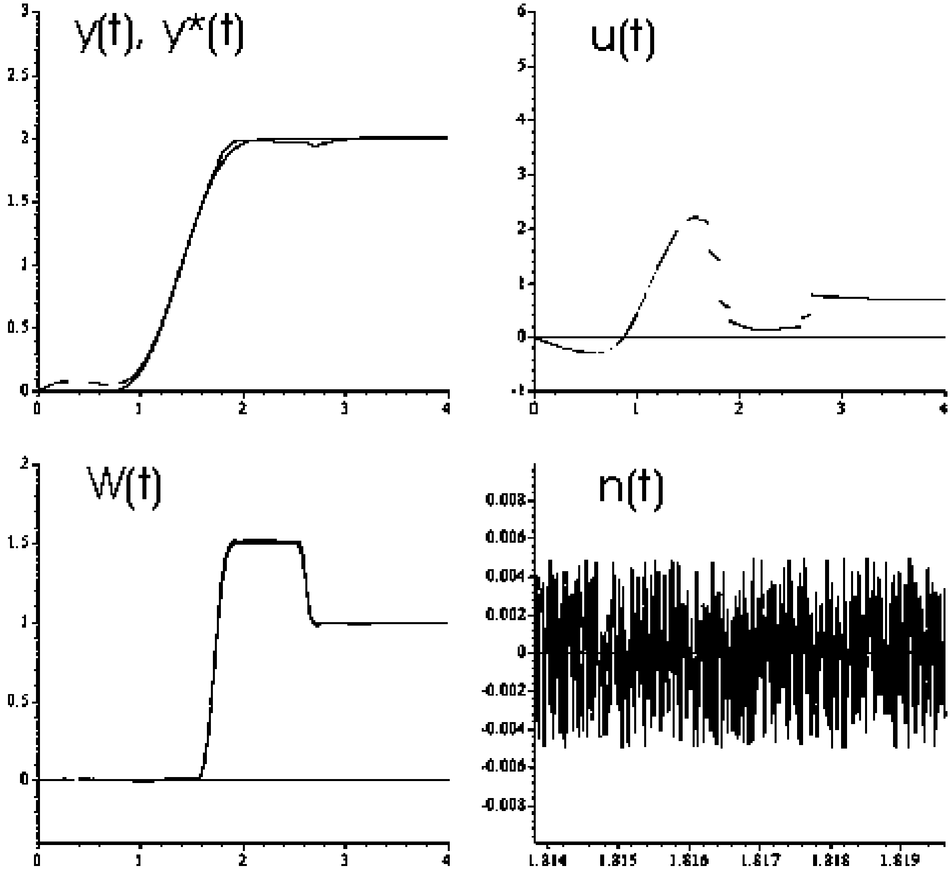
Identifying unknown failures

The feedback control can now be computed in an accommodating fashion

$$u = -\mathbf{w}_e(t) + \frac{1}{K}\left[\dot{y} + \tau\left(\dot{y}^* - k_2(y - y^*) - k_1\int_0^t (y - y^*(\sigma))d\sigma\right)\right]$$

For the implementation of this scheme, we compute the required successive time derivatives of the signal $y(t)$ and that of the signal $u(t)$ in an algebraic fashion.

A simple example



A not so simple example

Consider again the simple first order system

$$\begin{aligned} \tau \dot{x} &= -x + K[u + \mathbf{w}(t)] + \vartheta + n(t), \\ y &= x + \zeta(t) \end{aligned}$$

with τ being an unknown parameter. The failure $\mathbf{w}(t)$ is known to be piece-wise constant, ϑ is a permanent unknown constant perturbation and $n(t)$ and $\zeta(t)$ are colored zero mean random processes.

It is desired to track a given reference trajectory $y^*(t)$, identify the failure $\mathbf{w}(t)$, as well as the unknown parameters: ϑ and τ, while the system is operating in closed loop.

A not so simple example

We perform a fast identification for the unknown parameters ϑ and τ, fix them to their estimated values ϑ_e, τ_e and proceed to identify the up-coming failure signal $\mathbf{w}(t)$ using the following nonlinear parity equation.

$$\frac{1}{2}\tau_e\left[\frac{d}{dt}x^2\right] = -x^2 + Kux + K\mathbf{w}(t)x + \vartheta_e x$$

A not so simple example

We obtain the following failure estimator for $\mathbf{w}(t)$, which is updated every T units of time, at the instants t_i with a small calculation interval of length $\epsilon > 0$ at the re-initialization times t_i.

$$\mathbf{w}(t) = \begin{cases} w_e(t_i^-) & \text{for } t \in [t_i, t_i + \epsilon] \\ \frac{\frac{1}{2}\tau_e\left[(t-t_i)y^2\right]+z(t)}{\int_{t_i}^{t}(\sigma-t_i)x(\sigma)d\sigma} & \text{for } t \in [t_i + \epsilon, t_i + T] \end{cases}$$

with

$$\dot{z} = -\frac{1}{2}\tau_e y^2(t) - (t-t_i)\left[-y^2(t) + Ku(t)y(t) + \vartheta_e y(t)\right], \quad z(t_i) = 0$$

A not so simple example

The fast (algebraic) parameter identification scheme is performed while the failure, $\mathbf{w}(t)$, is zero.
It is carried out according to the techniques in Fliess-Sira-Ramírez CRASP 2002.

$$\begin{bmatrix} \tau_e \\ \vartheta_e \end{bmatrix} = \begin{cases} \begin{bmatrix} \tau_e(t_i^-) \\ \vartheta_e(t_i^-) \end{bmatrix} \text{ for } t \in [t_i, t_i + \epsilon] \\ \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}^{-1} \begin{bmatrix} \int_{t_i}^t (\sigma - t_i)\,[Ku(\sigma) - y(\sigma)]\,d\sigma \\ \int_{t_i}^t \int_{t_i}^\sigma (\lambda - t_i)\,[Ku(\lambda) - y(\lambda)]\,d\lambda d\sigma \end{bmatrix} \\ \text{for } t \in [t_i + \epsilon, t_i + T] \end{cases}$$

with

$$a_{11}(t) = (t - t_i)y - \int_{t_i}^t (\sigma - t_i)y \quad a_{12}(t) = (t - t_i)^2/2$$
$$\frac{d}{dt} a_{21}(t) = a_{11}(t), \;\; a_{21}(t_i) = 0 \qquad \frac{d}{dt} a_{22}(t) = a_{12}(t), \;\; a_{22}(t_i) = 0$$

A not so simple example

The accommodating controller is computed as a PI controller

$$\begin{aligned} u &= -\mathbf{w}_e(t) + \frac{1}{K}(y - \vartheta_e) \\ &\quad + \frac{\tau_e}{K}\left[\dot{y}^*(t) - k_2(y - y^*(t)) - k_1 \int_0^t (y - y^*(\sigma))d\sigma\right] \end{aligned}$$

In the computer simulations we used

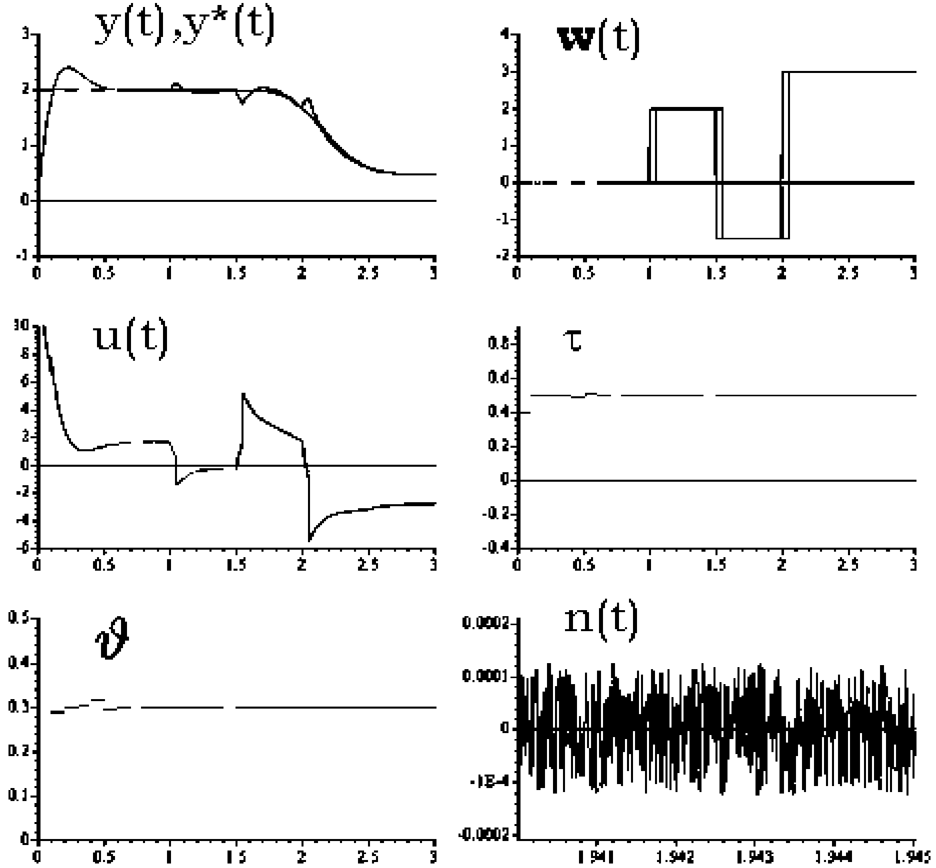
$$\tau = 0.5, \quad K = 1, \quad \vartheta = 0.3$$

$$\mathbf{w}(t) = 2 \times \mathbf{1}(t-1) - 3.5 \times \mathbf{1}(t-1.5) + 4.5 \times \mathbf{1}(t-2)$$

The random processes $n(t)$ and $\zeta(t)$ were chosen to be

$$n(t) = 10\zeta(t), \quad \zeta(t) = 2.5 \times 10^{-4}[\mathrm{rect}(t) - 0.5]$$

A not so simple example



DC motor example

Consider the following model of a separately excited DC motor

$$
\begin{aligned}
L_r \frac{dI_r}{dt} &= -R_r I_r - c f(I_s)\omega + u_r \\
\left(\frac{\partial f}{\partial I_s}\right) \frac{dI_s}{dt} &= -R_s I_s + u_s \\
J \frac{d\omega}{dt} &= c f(I_s) I_r - B\omega - \tau
\end{aligned}
$$

where:

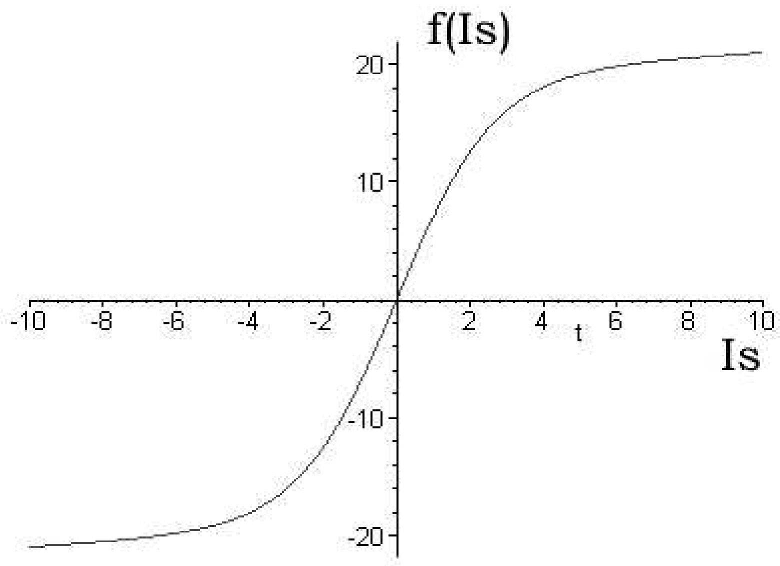
I_r: rotor current

I_s: stator current, $f(I_s)$: magnetization function

ω: angular velocity

τ: unknown *piecewise* constant load torque

DC motor example



DC motor example

The torque τ is modelled as a piecewise constant perturbation input first appearing at an unknown instant of time and, possibly, randomly changing amplitude values at unpredictable moments.

We think of the load torque, τ, as a *system failure* that needs to be *detected, identified and accommodated for*.

We will produce an accurate failure estimation, $\widehat{\tau}$, which is also *robust* with respect to system and measurement noises, by using nonlinear algebraic estimation techniques while the system is *operating in closed loop*.

DC motor example

The system is *flat* with flat outputs given by I_s and ω. Indeed, all variables are differentially parameterizable in terms of $F = \omega$ and $R = I_s$,

$$
\begin{aligned}
I_r &= \frac{J\dot{F} + BF}{cf(R)} \\
u_s &= \left(\frac{\partial f(R)}{\partial R}\right)\dot{R} + R_s R \\
u_r &= \frac{L_r}{c}\left[\frac{(J\ddot{F} + B\dot{F})f(R) - (J\dot{F} + BF)\frac{\partial f(R)}{\partial R}\dot{R}}{f^2(R)}\right] \\
&\quad + R_r\left(\frac{J\dot{F} + BF}{cf(R)}\right) + cf(R)F
\end{aligned}
$$

DC motor example

It is desired to track a given reference trajectory for the angular velocity, denoted by $\omega^*(t)$, and a given reference trajectory for the stator current, denoted by $I_s^*(t)$.
If the load torque perturbation, τ, were perfectly known, we would propose the following flatness based controller:

$$\begin{aligned} u_r &= \frac{L_r}{c}\left[\frac{(Jv + B\dot{\omega})f(I_s) - (J\dot{\omega} + B\omega + \tau)\frac{\partial f}{\partial I_s}z}{f^2(I_s)}\right] \\ &\quad + R_r\left(\frac{J\dot{\omega} + B\omega + \tau}{cf(I_s)}\right) + cf(I_s)\omega \\ u_s &= \left(\frac{\partial f}{\partial I_s}\right)z + R_s I_s \end{aligned}$$

where

$$\dot{\omega} = J^{-1}\left[cf(I_s)I_r - B\omega - \tau\right]$$

DC motor example

where the auxiliary control inputs v and z would be given by:

$$\begin{aligned} z &= \dot{I}_s^* - k_s(I_s - I_s^*(t)) \\ v &= \ddot{\omega}^* - k_{2r}(\dot{\omega} - \dot{\omega}^*) - k_{1r}(\omega - \omega^*) - k_{0r}\int_0^t (\omega - \omega^*(\sigma))d\sigma \end{aligned}$$

with

$$\dot{\omega} = J^{-1}\left[cf(I_s)I_r - B\omega - \tau\right]$$

The set of gains $\{k_s, k_{2r}, k_{1r}, k_{0r}\}$ would be chosen so that the decoupled chain of integrators exhibita stable poles in closed loop.

DC motor example

The tracking of the flat outputs reference trajectories would require the knowledge of the *unknown load torque* τ.

Consider the energy balance of the system relating the stored energy, the dissipated energy and the input power

$$\frac{d}{dt}\left[L_r I_r^2 + J\omega^2\right] + R_r I_r^2 + B\omega^2 = u_r I_r - \tau\omega$$

Using this physically meaningful *nonlinear parity equation*, we obtain, by means of elementary algebraic manipulations, the following estimate of the unknown load torque as a function of the measurable states ω, I_r and the control input u_r.

DC motor example

$$\widehat{\tau} = \begin{cases} \frac{n(t)}{d(t)} & \text{for } t \geq t_i + \epsilon \\ n(t) = 0.5(t-t_i)(L_r I_r^2 + J\omega^2) & \\ + \int_{t_i}^{t} \Big[(\sigma - t_i)(R_r I_r^2 + B\omega^2 - u_r I_r) & \\ -0.5(L_r I_r^2 + J\omega^2) \Big] d\sigma & \\ d(t) = \int_{t_i}^{t} (\sigma - t_i)\omega(\sigma) d\sigma & \\ \widehat{\tau}(t_i^-) & \text{for } t \in [t_i, t_i + \epsilon) \end{cases}$$

$$t_i = kT, \; k = 0, 1, 2, ..., \; T > 0.$$

The formula depicts periodic resettings at the end of intervals of duration T and the provision of a small calculation time $\epsilon > 0$.

DC motor example

We propose the following flatness based controller with accommodation for the load torque in terms of the algebraic estimate, $\widehat{\tau}$, for the input load torque τ.

$$
\begin{aligned}
u_r &= \frac{L_r}{c}\left[\frac{(Jv + B\widehat{\dot{\omega}})f(I_s) - (J\widehat{\dot{\omega}} + B\omega + \widehat{\tau})\frac{\partial f}{\partial I_s}z}{f^2(I_s)}\right] \\
&\quad + R_r\left(\frac{J\widehat{\dot{\omega}} + B\omega + \widehat{\tau}}{cf(I_s)}\right) + cf(I_s)\omega \\
u_s &= \left(\frac{\partial f}{\partial I_s}\right)z + R_s I_s
\end{aligned}
$$

where

$$\widehat{\dot{\omega}} = J^{-1}\left[cf(I_s)I_r - B\omega - \widehat{\tau}\right]$$

DC motor example

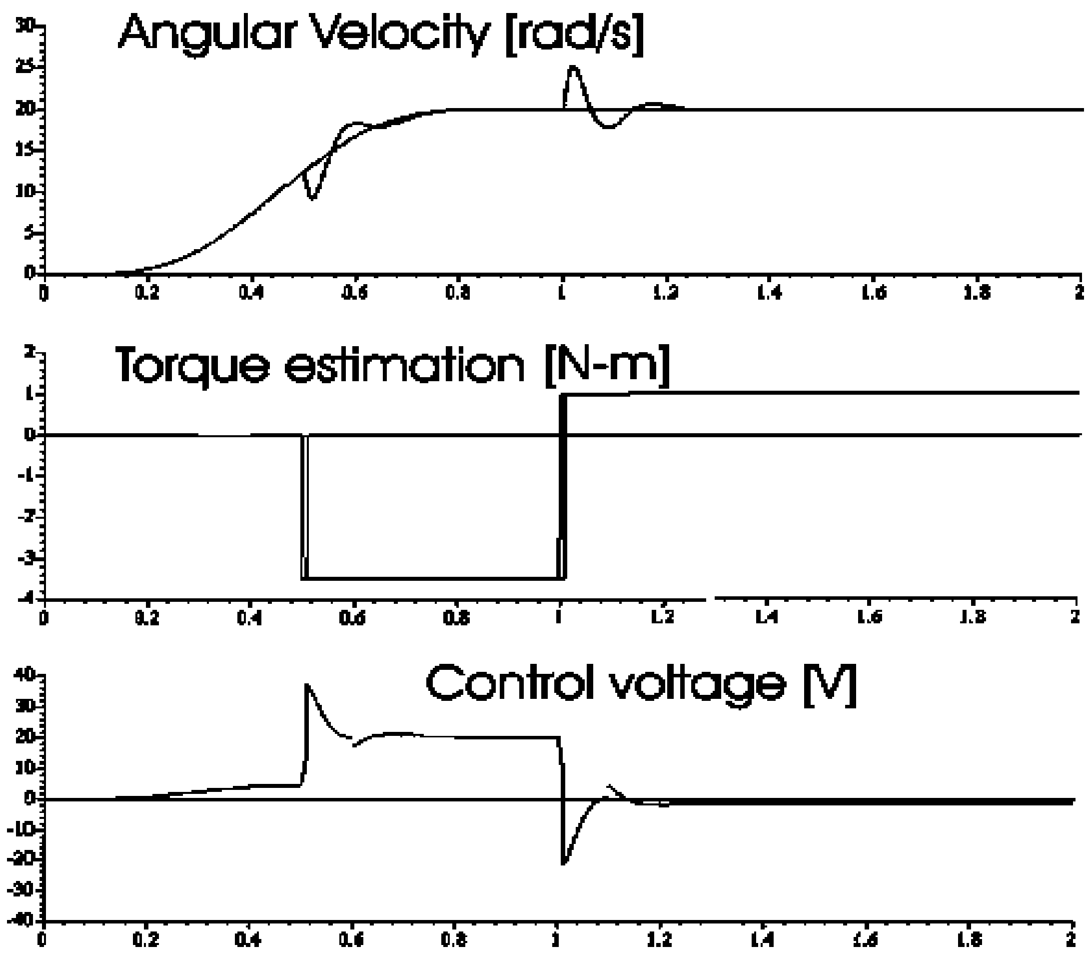
where the auxiliary control inputs v and z are given by:

$$
\begin{aligned}
z &= \dot{I}_s^* - k_s(I_s - I_s^*(t)) \\
v &= \ddot{\omega}^* - k_{2r}(\widehat{\dot{\omega}} - \dot{\omega}^*) - k_{1r}(\omega - \omega^*) - k_{0r}\int_0^t (\omega - \omega^*(\sigma))d\sigma
\end{aligned}
$$

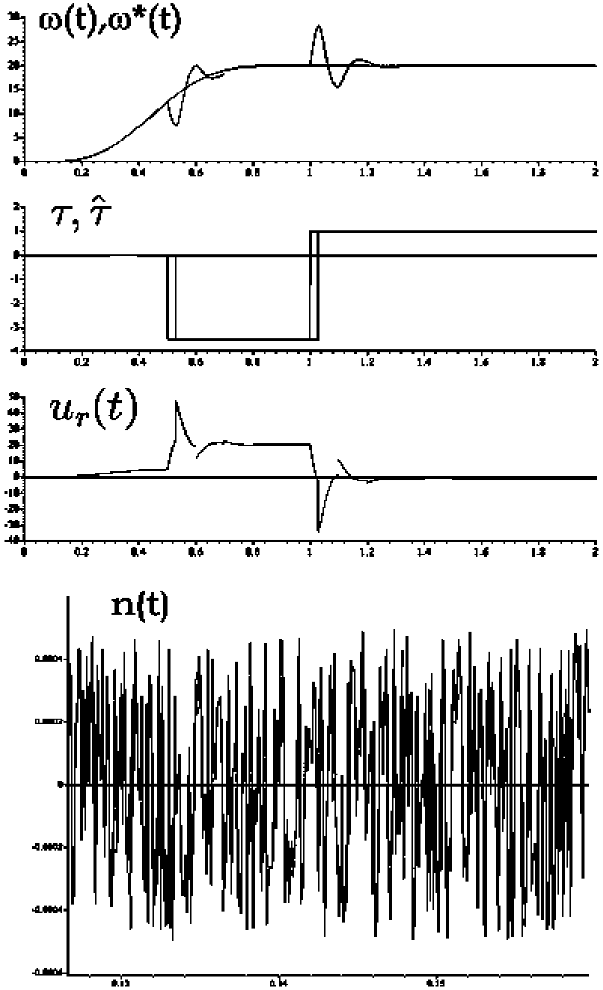
with

$$
\widehat{\dot{\omega}} = J^{-1}\left[cf(I_s)I_r - B\omega - \widehat{\tau}\right]
$$

Simulations



Simulations



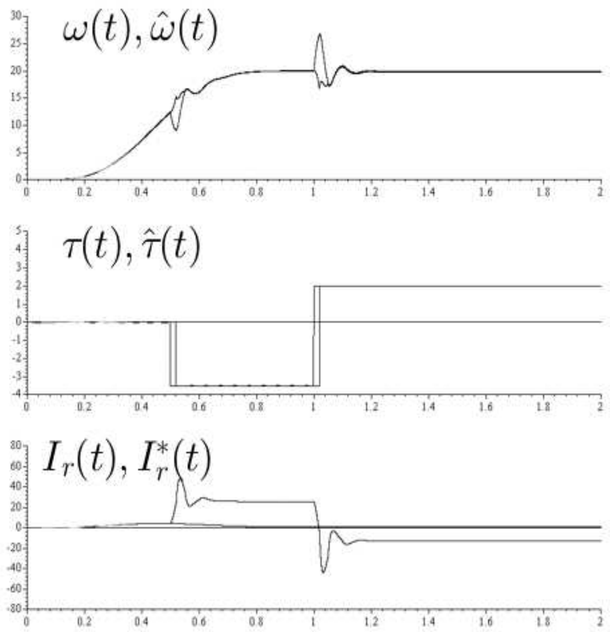
A sensorless scheme

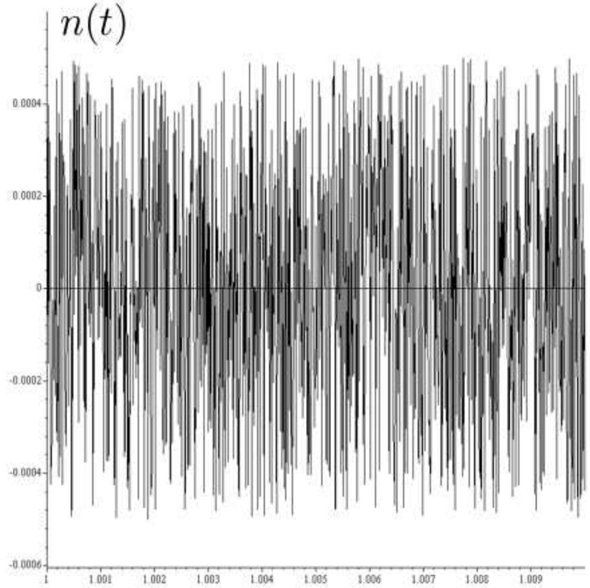
A *sensorless* scheme which implies the measurement of only the stator and rotor currents may be achieved by estimating the angular velocity in a GPI fashion (structural estimate) with the help of the rotor current equation and an integral compensation term for the computational mismatch in the torque accommodation.

$$\hat{\omega} = J^{-1}\int_0^t \left[cf(I_s)I_r - \widehat{\tau} - B\omega\right] ds$$

$$\omega = \frac{u_r - L_r\frac{dI_r}{dt} - R_rI_r}{cf(I_s)}$$

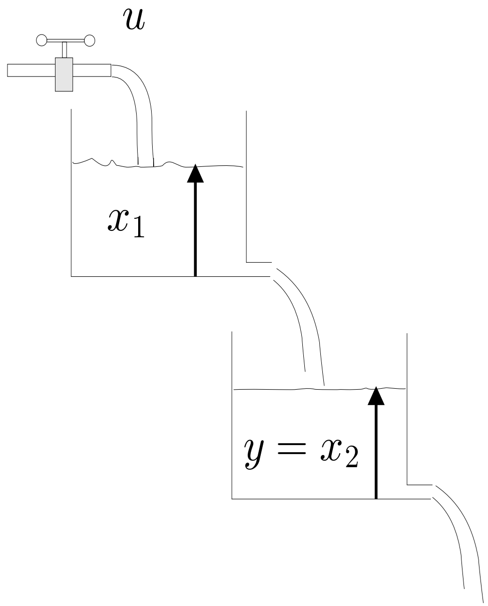
Simulations





A water tank example

Consider the frollowing tank system:



The description of the system is given by

$$\begin{aligned} \dot{x}_1 &= -c\sqrt{x_1} + u \\ \dot{x}_2 &= c\sqrt{x_1} - c\sqrt{x_2} \\ y &= x_2 \end{aligned}$$

A water tank example

In this system,

$$\begin{aligned}\dot{x}_1 &= -c\sqrt{x_1} + u + \mathbf{w}(t) + \varpi \\ \dot{x}_2 &= c\sqrt{x_1} - c\sqrt{x_2} \\ y &= x_2\end{aligned}$$

- The perturbation ϖ is constant but unknown
- The actuator failure $\mathbf{w}(t)$ is of unknown nature
- Only $y = x_2$ is available for measurement
- It is desired to track a given trajectory $y^*(t)$.
- The actuator failure starts at some time $t >> 0$
- The constant c is known

A water tank example

The water tank system is *flat* with flat output given by $y = x_2$.
We obtain the following perturbed parameterization of the system variables in terms of the flat output y,

$$
\begin{aligned}
x_1 &= \frac{1}{c^2}\left(\dot{y} + c\sqrt{y}\right)^2 \\
u &= -\mathbf{w}(t) - \varpi + \left(\dot{y} + c\sqrt{y}\right)\left[\frac{2}{c^2}\left(\ddot{y} + c\frac{\dot{y}}{2\sqrt{y}}\right) + 1\right]
\end{aligned}
$$

A water tank example

An accommodating linearizing feedback controller may be established on the basis of the estimates of the perturbation inputs $\hat{\mathbf{w}}(t)$, $\hat{\varpi}$.

$$
\begin{aligned}
u &= -\hat{\mathbf{w}}(t) - \hat{\varpi} + (\dot{y} + c\sqrt{y})\left[\frac{2}{c^2}\left(v + c\frac{\dot{y}}{2\sqrt{y}}\right) + 1\right] \\
v &= \ddot{y}^* - k_2(\dot{y} - \dot{y}^*) - k_1(y - y^*) - k_0\int_0^t (y - y^*(\sigma))d\sigma
\end{aligned}
$$

Wherever $\dot{y}$ is required, we obtain it directly from our algebraic mehthod for derivative calculation, on the basis of the measured output y.

A water tank example

The estimation of the constant perturbation ϖ is readily accomplshed from the equation governing the system before the failure

$$\dot{x}_1 = -c\sqrt{x_1} + u + \varpi$$

multiplying by t and integrating by parts we obtain the following estimate of ϖ

$$\hat{\varpi} = \begin{cases} \text{arbitrary} & t \in [0, \epsilon) \\ \frac{t\hat{x}_1 - \int_0^t \left[\hat{x}_1 - \sigma(c\sqrt{\hat{x}_1(\sigma)} - u(\sigma))\right] d\sigma}{t^2/2} & t \in [\epsilon, \infty) \end{cases}$$

with

$$\hat{x_1} = \frac{1}{c^2}\left(\dot{y} + c\sqrt{y}\right)^2$$

where the time derivative of y is computed by algebraic means.

A water tank example

The estimation of the unknown signal $\mathbf{w}(t)$, under the assumption that $\mathbf{w}(0) = 0$, is as follows:

• Obtain the input-output relation:

$$\frac{2}{c^2}\left(\dot{y} + c\sqrt{y}\right)\left(\ddot{y} + c\frac{\dot{y}}{2\sqrt{y}}\right) + \left(\dot{y} + c\sqrt{y}\right) = u + \mathbf{w}(t) + \varpi$$

• Take one time derivative in order to annihilate ϖ and avoid an algebraic loop at the accommodated feedback controller design stage:

$$\frac{2}{c^2}\left\{\left[y^{(3)} + \frac{c}{4}\left(\frac{2\ddot{y}y - \dot{y}^2}{y\sqrt{y}}\right)\right]\left(\dot{y} + c\sqrt{y}\right) + \left(\ddot{y} + \frac{c}{2}\frac{\dot{y}}{\sqrt{y}}\right)^2\right\} + \ddot{y} + \frac{c}{2}\left(\frac{\dot{y}}{\sqrt{y}}\right) = \dot{u} + \dot{\mathbf{w}}(t)$$

• Obtain $\mathbf{w}(t)$ by integrating $\dot{\mathbf{w}}(t)$ form the last expression with the derivatives of y and u. Generating as many derivatives as needed.

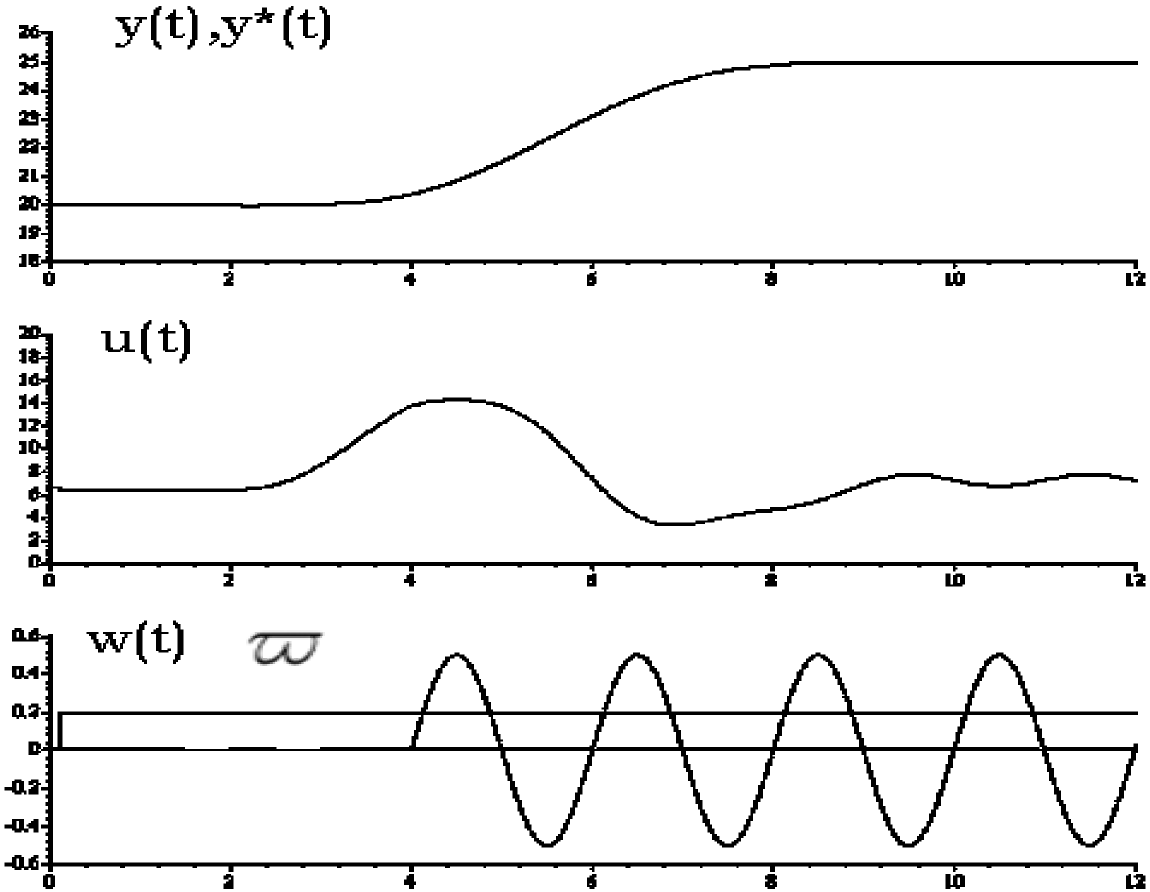
A water tank example

$$
\begin{aligned}
\mathbf{w}(t) &= \int_0^t \Bigg\{ -\dot{u} + \frac{2}{c^2}\left[\left(y^{(3)} + \frac{c}{4}\left(\frac{2\ddot{y}y - \dot{y}^2}{y\sqrt{y}}\right)\right)(\dot{y} + c\sqrt{y}) + \left(\ddot{y} + \frac{c}{2}\frac{\dot{y}}{\sqrt{y}}\right)^2\right] \\
&\qquad + \ddot{y} + \frac{c}{2}\left(\frac{\dot{y}}{\sqrt{y}}\right)\Bigg\} d\sigma
\end{aligned}
$$

where, evidently we now have: $\mathbf{w}(0) = 0$.

A slightly computationally involved but, certainly, not a more difficult problem, arises when the constant c is also unknown.

A water tank example



Appendix: Derivatives calculation

Consider a truncated Taylor series expansion model for the polynomial approximation of a given signal $y(t)$ which is assumed to be sufficiently smooth.

$$\widetilde{y}(t) = \sum_{k=0}^{N-1} \frac{1}{k!} y^{(k)}(t_i)(t - t_i)^k$$

This approximation satisfies the homogeneous linear, time-invariant, differential equation:

$$\frac{d^N \widetilde{y}}{dt^N} = 0$$

The local derivative estimation problem is then reduced to compute the states of a time-invariant, linear, homogeneous system of order N.

Derivatives calculation

The adopted linear approximation $\widetilde{y}^{(N)}(t) = 0$ satisfies, in terms of operational transforms, the following relation ($\widetilde{y}$ replaced by y):

$$\frac{d^N}{ds^N}\left[s^N Y(s)\right] = 0$$

The expressions given by

$$s^{-k}\frac{d^N}{ds^N}\left[s^N Y(s)\right] = 0, \quad k = N-1, N-2, \cdots, N-k$$

contain, respectively, implicit information on the first, second, etc., k-th derivatives of $y(t)$ in an approximate manner.

Example

Consider a fifth order approximation in $t = 0$ of a sufficiently differentiable signal, $y(t)$:

$$\widetilde{y}(t) = \sum_{k=0}^{5} \frac{1}{k!} t^k y^{(k)}(0)$$

The proposed formula is written as follows:

$$s^{-k} \frac{d^6}{ds^6}[s^6 Y(s)] = 0, \quad k = 5, 4, 3$$

Example

$$\left[720s^{-5}Y(s) + 4320s^{-4}\frac{dY(s)}{ds} + 5400s^{-3}\frac{d^2Y(s)}{ds^2}\right.$$
$$+2400s^{-2}\frac{d^3Y(s)}{ds^3} + 450s^{-1}\frac{d^4Y(s)}{ds} + 36\frac{d^5Y(s)}{ds^5}$$
$$\left.+s\frac{d^6Y(s)}{ds^6}\right] = 0$$

This expression relates the term $s\frac{d^6Y(s)}{ds^6}$, which in the time domain is just: $\frac{d}{dt}[t^6y(t)] = 6t^5y(t) + t^6\dot{y}(t)$, with a finite sum of convolutions of the signal $y(t)$ with powers of t.

Example

$$\Big[720(\int^{(5)} y(t)) - 4320(\int^{(4)} ty(t)) + 5400(\int^{(3)} t^2 y(t))$$

$$-2400(\int^{(2)} t^3 y(t)) + 450 \int t^4 y(t) - 36t^5 y(t)$$

$$+\frac{d}{dt} t^6 y(t)\Big] = 0$$

where

$$(\int^{(k)} t^j y(t)) = \int_0^t \int_0^{\sigma_1} \cdots \int_0^{\sigma_{k-1}} \sigma_k^j y(\sigma_k) d\sigma_k \cdots d\sigma_1$$

Example

$$
\begin{aligned}
\dot{y}(t) \quad = \quad & \frac{1}{t^6}\Big[-720\Big(\int^{(5)} y(t)\Big) + 4320\Big(\int^{(4)} ty(t)\Big) \\
& -5400\Big(\int^{(3)} t^2 y(t)\Big) + 2400\Big(\int^{(2)} t^3 y(t)\Big) - 450\int t^4 y(t) \\
& +30t^5 y(t)\Big]
\end{aligned}
$$

The obtained formula presents a singularity at $t = 0$, which disappears for any $t = \epsilon > 0$.

Example

We propose the following estimate of the first order time derivative of $y(t)$ with respect to time:

$$\dot{y}(t) = \begin{cases} arbitrary\ constant & 0 \leq t < \epsilon \\ \frac{1}{t^6}\Big[-720(\int^{(5)} y(t)) + 4320(\int^{(4)} ty(t)) & \\ -5400(\int^{(3)} t^2 y(t)) + 2400(\int^{(2)} t^3 y(t)) & \\ -450 \int t^4 y(t) + 30t^5 y(t)\Big] & t \geq \epsilon \end{cases}$$

Example

This computation can be expressed in the form of a time varying linear filter of the form:

$$\dot{y}(t) = \begin{cases} arbitrary\ constant & 0 \leq t < \epsilon \\ \frac{1}{t^6}\left[30t^5 + z_1\right] & t \geq \epsilon \end{cases}$$

where

$$\begin{aligned} \dot{z}_1 &= z_2 - 450t^4 y(t) \\ \dot{z}_2 &= z_3 + 2400t^3 y(t) \\ \dot{z}_3 &= z_4 - 5400t^2 y(t) \\ \dot{z}_4 &= z_5 + 4320t y(t) \\ \dot{z}_5 &= -720 y(t) \end{aligned}$$

Example

To compute the second order time derivative *we integrate one time less* the originally proposed expression

$$\Big[720s^{-4}Y(s) + 4320s^{-3}\frac{dY(s)}{ds} + 5400s^{-2}\frac{d^2Y(s)}{ds^2}$$
$$+2400s^{-1}\frac{d^3Y(s)}{ds^3} + 450\frac{d^4Y(s)}{ds} + 36s\frac{d^5Y(s)}{ds^5}$$
$$+s^2\frac{d^6Y(s)}{ds^6}\Big] = 0$$

The last two terms in the sum are written as $-36\frac{d}{dt}[t^5y(t)] + \frac{d^2}{dt^2}[t^6y(t)]$ which allows one to compute the second order time derivative of $y(t)$ in terms of the first order time derivative and convolutions of $y(t)$.... and so on and so forth...

Example

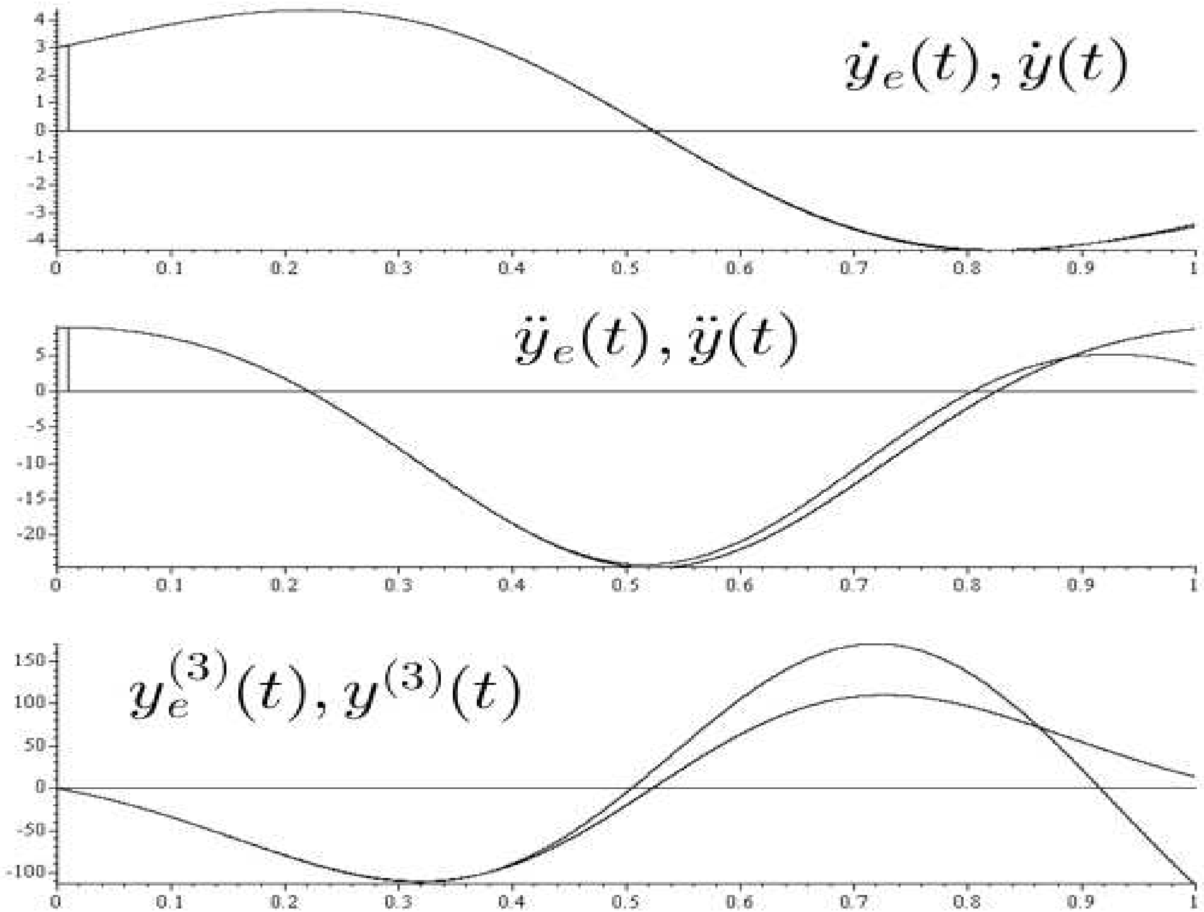
In the time domain we obtain

$$\ddot{y} = \frac{1}{t_6}\left(300t^4 y(t) - 24t^5 \dot{y}(t) - z_2\right)$$

$$\begin{aligned}
\dot{z}_1 &= z_2 - 450t^4 y(t) \\
\dot{z}_2 &= z_3 + 2400t^3 y(t) \\
\dot{z}_3 &= z_4 - 5400t^2 y(t) \\
\dot{z}_4 &= z_5 + 4320t y(t) \\
\dot{z}_5 &= -720 y(t)
\end{aligned}$$

Example

$$y(t) = e^{sin(\omega t)}, \quad \omega = 3$$



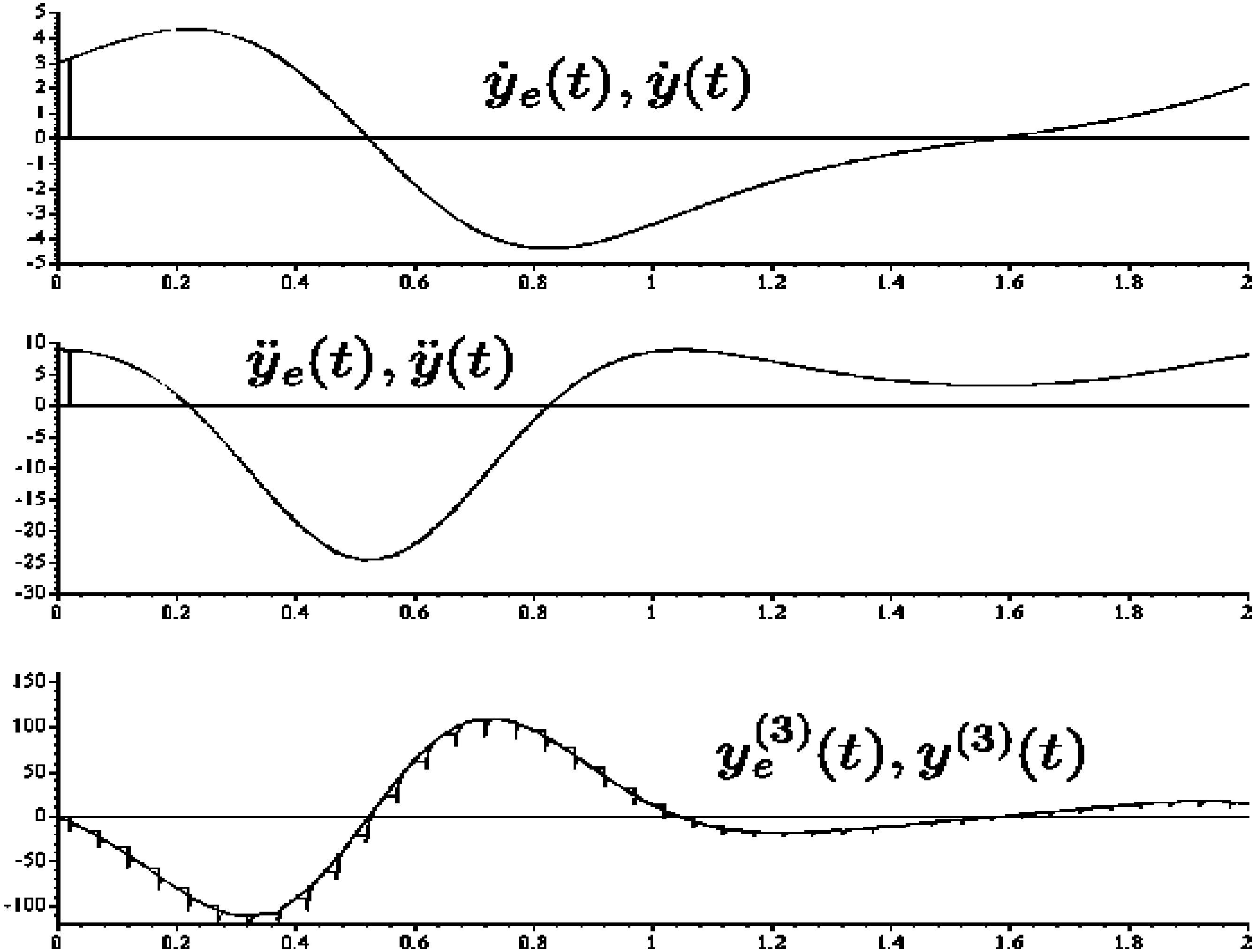
Remark

The validity of the formulae for the calculation of the time derivatives is limited in the time horizon. For this reason, it becomes necessary to *re-initialize* the computations at some time $t_r > 0$.

As the derivatives drift from their actual values, so will the estimated signal computed on the basis of the truncated Taylor series approximation and the estimated values of the signal's time derivative.

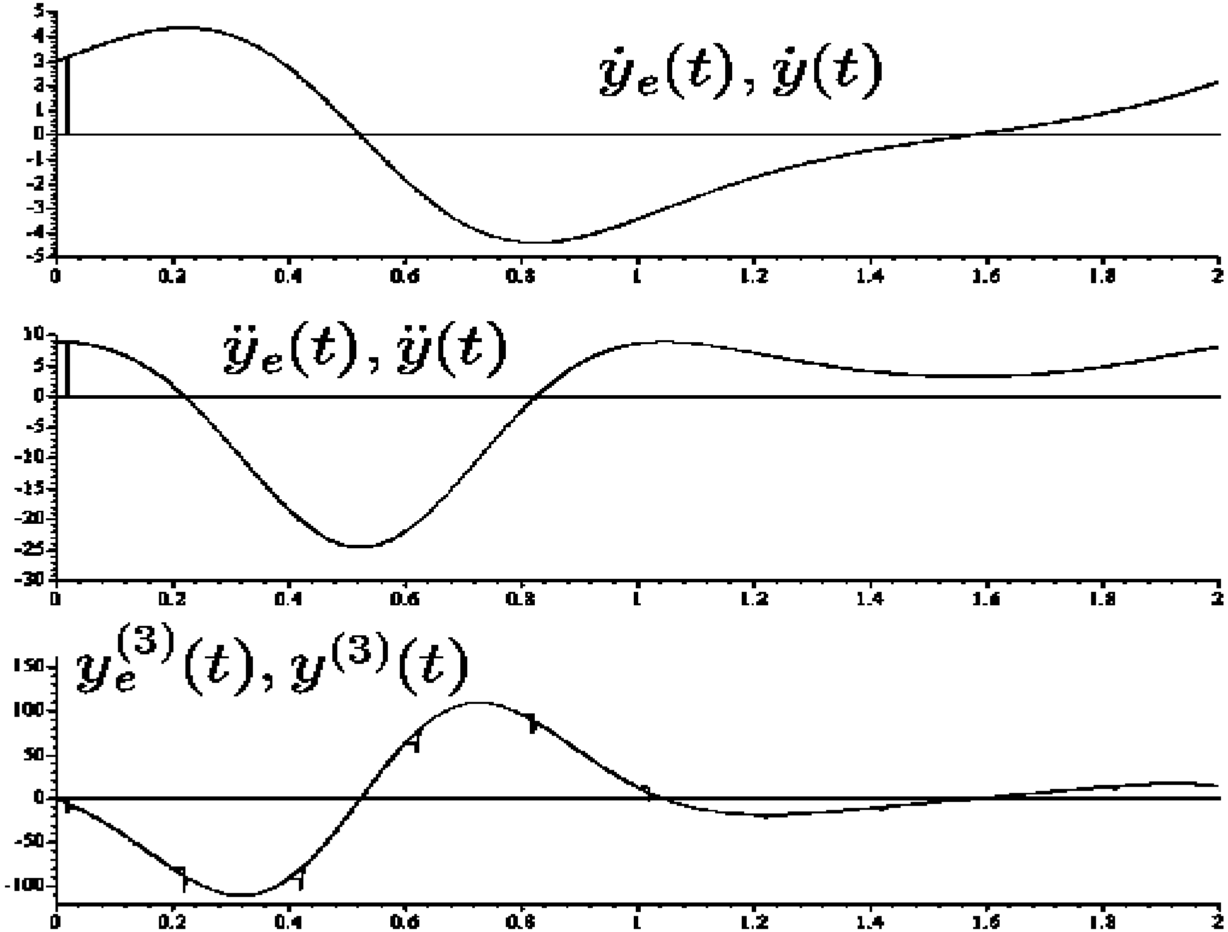
Example

$$\epsilon = 0.02, \quad T = 0.05$$



Example

$$\epsilon = 0.02, \quad T = 0.2$$



Remark

An automatic resetting of the calculations can be devised on the basis of the integral square error of the reconstructed signal deviation and a pre-specified threshold value for such a reconstruction error,

$$e = \int_{t_r+\epsilon}^{t} |y(\sigma) - \widehat{y}(\sigma)|^2 d\sigma$$

$$\widehat{y}(t) = y(t_r+\epsilon)+[\dot{y}(t_r+\epsilon)](t-t_r-\epsilon)+\frac{1}{2}[\ddot{y}(t_r+\epsilon)](t-t_r-\epsilon)^2+\cdots$$

or

$$\widehat{y}(t) = y(t_r + \epsilon) + \int_{t_r+\epsilon}^{t} [\dot{y}]_e(\sigma) d\sigma$$

Conclusions

- The algebraic approach, recently proposed for fault detection and parameter identification in time-invariant linear systems, may be suitably extended to nonlinear dynamic systems.
- A crucial step in fault tolerant schemes for nonlinear systems is that of the state estimation that, as we have shown here, may be easily achieved in terms of input and output derivative calculation. These procedures are also using algebraic techniques.
- The efficent computation of time derivatives allows for the use of input output models in nonlinear systems for perturbation identification, parameter calculations and the gathering of failure data for robust feedback control design purposes.

Conclusions

- The algebraic treatment of many problems in automatic control, such as state estimation, perturbation identification and cancellation, as well as problems in other areas such as: comunication systems and signal processing (signal compression and image analyisis) greatly simplifies the mathematical attack on most of the important problems in these fields and it yields computationally implementable schemes, or algorithms characterized by fast computations.