Robust Fault detection for uncertain switched systems: $\mathcal{H}_\infty$ approach

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Motivations

Passive security systems
- Airbag
- Security belt

Active security and comfort systems
- ABS: Anti-lock braking system
- ESP: Electronic stability program
- ACC: Autonomous cruise control
- Suspension control
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Problematic
What happens in case of sensors/actuators failure?
Could we avoid dramatic situation by detecting/estimating these faults?
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Outline

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   • Backgrounds

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   • Formulations

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5 Conclusion
A common fault detection technique is to calculate a residual signal.

In this study, we consider the model based techniques, and a focus on $\mathcal{H}_\infty$ fault detection problem.
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For some problems, when fault frequency range is known (low frequencies, offsets...), the fault detection problem can be transformed into fault estimation problem.
A common fault detection technique is to calculate a residual signal.

In this study, we consider the model based techniques, and a focus on $\mathcal{H}_\infty$ fault detection problem.

For some problems, when fault frequency range is known (low frequencies, offsets...), the fault detection problem can be transformed into fault estimation problem.

Switched uncertain approach may offer an interesting framework to model non-linear plants with parameters variation.

Working in the switched uncertain system framework can be used for the FD problem in lateral vehicle control.
Multi-objective design

A reliable fault detection filter must meet several specification:

- (a) perturbation (and unknown inputs, noises) rejection,
- (b) sensitivity toward faults,
- (c) and robustness toward uncertainties
- (d) correct time response for fault detection/estimation.
Multi-objective design

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In our approach, a discrete time fault detection and estimation filter design is developed. The steps of the design are as follows:
- Express an $\mathcal{H}_\infty$ performance, useful to ensure the residual robustness to unknown inputs for switched systems.
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In our approach, a discrete time fault detection and estimation filter design is developed. The steps of the design are as follows:

- Express an $\mathcal{H}_\infty$ performance, useful to ensure the residual robustness to unknown inputs for switched systems.
- Formulate the problem into Bilinear Matrix Inequalities (BMI), then linearization using projection lemma, while introducing several degrees of freedom.
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- Formulate the problem into Bilinear Matrix Inequalities (BMI), then linearization using projection lemma, while introducing several degrees of freedom.
- Extend the proposed method to consider the uncertainties in the system.
- Finally, add loop shaping via weighting filters, useful to tune the transient response and steady state response, and enhance the overall fault detection.
Problem formulation
A linear switched system

A state space representation of switched linear time invariant system is:

\[
\begin{align*}
    x_{k+1} &= A_{\alpha(k)} x_k + E_{d,\alpha(k)} d_k + E_{f,\alpha(k)} f_k \\
    y_k &= C_{\alpha(k)} x_k + F_{d,\alpha(k)} d_k + F_{f,\alpha(k)} f_k
\end{align*}
\]  

(1)

- \( x \in \mathbb{R}^n \) is the state vector,
- \( y \in \mathbb{R}^p \) is the measurement output vector,
- \( \alpha(k) \) is the switching signal
- \( d \in \mathbb{R}^{n_d} \) is the disturbance vector,
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The switching signal is assumed unknown a priori but its value is real-time available.
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**Assumption 2**

The pairs \((A_{\alpha}, C_{\alpha})\) are assumed observable, or without loss of generality are detectable. It is a standard assumption for all fault detection problems.
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**Notation**

When \(\alpha(k) = i\), it means that the \(i^{th}\) subsystem is activated. Moreover, at the switching time \(k : i = \alpha(k) \neq \alpha(k+1) = j\).
Consider the state space representation of uncertain switched system:

\[
\begin{align*}
    x_{k+1} &= \bar{A}_{\alpha(k)} x_k + E_{d,\alpha(k)} d_k + E_{f,\alpha(k)} f_k \\
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(2)

In this study, an additive uncertainties form is considered:

\[
\bar{A}_{\alpha(k)} = A_{\alpha(k)} + H_{A,\alpha(k)} \Delta_{A,\alpha(k)} N_{A,\alpha(k)}
\]  

(3)

with \(H_{A,\alpha(k)} \in \mathbb{R}^{n \times nh}\) and \(N_{A,\alpha(k)} \in \mathbb{R}^{n \times na}\)
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Assumption 3

The system (2) is assumed stable, or without loss of generality is stabilizable (in case of joint fault detection and control).
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**Assumption 4**

The state uncertainty matrix \( \Delta A_{\alpha} \) is assumed bounded:

\[
\Delta_{A,\alpha}^T \Delta A_{\alpha} \leq I
\]
Problem formulation

We propose to design a switched fault estimation filter \( K_{f,i}(q^{-1}) \) with the following realization:

\[
\begin{align*}
    x^K_{k+1} &= A^K_i x^K_k + B^K_i y_k \\
    \hat{f}_k &= C^K_i x^K_k + D^K_i y_k
\end{align*}
\]

(4)

where \( x^K \in \mathbb{R}^n_k \) is the filter’s state vector and \( \hat{f} \in \mathbb{R}^{n_f} \) is the vector of the estimated faults.

**Figure:** Fault estimation schemes
We propose to design a switched fault estimation filter $K_{f,i}(q^{-1})$ with the following realization:

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Define $z_k = f_k - \hat{f}_k$ and $x_k^a = [x_k^T \ x_k^{KT}]^T$ then:

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\end{cases}
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(5)

Where $A^a_i = \begin{bmatrix} A_i & 0 \\ B^K_i C_i & A^K_i \end{bmatrix}$, $E^a_i = \begin{bmatrix} E_{w,i} \\ B^K_i F_{w,i} \end{bmatrix}$, $C^a_i = [D^K_i C_i + \tilde{C}_i \quad C^K_i]$, $F^a_i = D^K_i F_{w,i} + \tilde{F}_i$, $\tilde{C}_i = [0]$, and $\tilde{F}_i = [0 \quad -I]$.  

Figure: Fault estimation schemes
Problem formulation

Define $T_{zw}(q^{-1})$ as the transfer from $w_k$ to the fault estimation error $z_k$:

$$T_{zw}(q^{-1}) = C_i(q^{-1}I - A_i)^{-1}E_i + F_i$$  \hspace{1cm} (6)
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The $L_2$-gain from the estimation error $z_k$ to the vector $w_k$ must be bounded by positive scalar $\gamma_i$ i.e.

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That is equivalent to the following inequality ($H_\infty$ norm):

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With $\gamma_i$ is positive scalars.
Define $T_{zw}(q^{-1})$ as the transfer from $w_k$ to the fault estimation error $z_k$:

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The problem is formulated as following: Find a discrete filter $K(q^{-1})$ such that the augmented system is stable and the equation (8) is satisfied.
Filter design
Theorem (Bounded real lemma for switched system)

For a given switched linear system under arbitrary switching, if there exist matrices $P_i$ and a positive definite matrices $P_i \forall i, j \in \{1..N\}$ such that:

\[
\begin{bmatrix}
-P_{i}^{-1} & A_i & E_i & 0 \\
A_i^T & -P_i & 0 & C_i^T \\
E_i^T & 0 & -\gamma_i^2 I & F_i^T \\
0 & C_i & F_i & -I
\end{bmatrix} < 0
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(9)

Then the switched discrete time fault estimation filter (SDTFEF) can be designed where the condition (8) is guaranteed.
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Proof

Using the switched Lyapunov function $V_k = x_k^{aT}P_kx_k^a > 0$ that must be decreasing for all $k$, solve $x_{k+1}^a P_{k+1} x_{k+1}^a - x_k^{aT}P_kx_k^a + z_k^T z_k - \gamma_i^2 w_k^T w_k < 0$. After some calculation and with Shur complement, (9) is easily obtained.
**Theorem (Bounded real lemma for switched system)**

For a given switched linear system under arbitrary switching, if there exist matrices $P_i$ and a positive definite matrices $P_i$ for all $i, j \in \{1..N\}$ such that:

$$\begin{bmatrix}
-P_i^{-1} & A_i^a & E_i^a & 0 \\
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**Non linearities**

This inequation is bilinear in $P_i$ and $P_j$. Linearization is needed.
On the structure of $P_i$ and BMI linearization

In design procedure involving switched system, numerous approach on the structure of $P_i$ are applied, we can cite:
On the structure of $P_i$ and BMI linearization

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1. Consider a commune Lyapunov function instead of switched one, i.e.: $P_i = P_j \ldots \forall i, j \in \{1..N\}$. This method is very conservative.
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3. In our study, no assumption on $P_i$ is made, we simply denote: First, $P_i$ is defined as follows:

$$P_i = \begin{bmatrix} S_i \\ N_i^T \end{bmatrix} \begin{bmatrix} N_i \\ U_i \end{bmatrix} = \begin{bmatrix} R_i & M_i \\ M_i^T & V_i \end{bmatrix}^{-1} > 0$$  \hspace{1cm} (10)
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On another hand, there are two methods to linearize the BMI problem:

Two approaches:
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On another hand, there are two methods to linearize the BMI problem:

Two approches:

1. Multiply the BMI by some full rank matrices with zeros, and then proceed to a change of variable
On the structure of $P_i$ and BMI linearization

In design procedure involving switched system, numerous approach on the structure of $P_i$ are applied, we can cite:

1. Consider a commune Lyapunov function instead of switched one, i.e.: $P_i = P_j... \forall i, j \in \{1..N\}$. This method is very conservative.

2. Impose some structure on $P$, for example: $P_i = \begin{bmatrix} P_{1,i} & 0 \\ 0 & P_{2,i} \end{bmatrix}$

3. In our study, no assumption on $P_i$ is made, we simply denote: First, $P_i$ is defined as follows:

$$P_i = \begin{bmatrix} S_i \\ N_i^T \end{bmatrix} \begin{bmatrix} N_i \\ U_i \end{bmatrix} = \begin{bmatrix} R_i & M_i \\ M_i^T & V_i \end{bmatrix}^{-1} > 0 \tag{10}$$

On another hand, there are two methods to linearize the BMI problem:

Two approches:

1. Multiply the BMI by some full rank matrices with zeros, and then proceed to a change of variable

2. Use the projection lemma to eliminate the unknown terms $K_i$, and then solve two sets of LMI, ones after another. This is explained in the next slides
Lemma (Projection lemma)

Given a symmetric matrix $\Psi$ and the matrices $N, \theta$ and $N$ with appropriate dimensions:

$$\Phi + M^T \theta^T N + N^T \theta M < 0$$

Denote by $W_X$ any matrices whose columns form bases of the null spaces of $X$. The above equation is solvable for $\theta$ if and only if $W_M^T \Psi W_M < 0$ and $W_N^T \Psi W_N < 0$
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Re-write (9) as:

$$\begin{bmatrix}
-P_j^{-1} & A_i^0 & E_i^0 & 0 \\
A_i^{0T} & -P_i & 0 & C_i^{0T} \\
E_i^{0T} & 0 & -\gamma_i^2 I & F_i^{0T} \\
0 & C_i & F_i & -I \\
\end{bmatrix} _{\Phi_j} + \begin{bmatrix}
B_i \\
0 \\
0 \\
D_i \\
\end{bmatrix} _{\Phi_j} \begin{bmatrix}
\mathcal{K}_i \\
0 \\
C_i \\
\mathcal{F}_i \\
0 \\
\end{bmatrix} _{\mathcal{K}_i} + [\ast] < 0 \quad (11)$$
Lemma (Projection lemma)

Given a symmetric matrix $\Psi$ and the matrices $N, \theta$ and $N$ with appropriate dimensions:

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Re-write (9) as:

$$\Phi_i + \begin{bmatrix} -P_j^{-1} & A_i^0 & E_i^0 & 0 \\ A_i^0 & -P_i & 0 & C_i^{0T} \\ E_i^0 & 0 & -\gamma_i^2 I & F_i^0 \\ 0 & C_i^0 & F_i^0 & -I \end{bmatrix} + \begin{bmatrix} B_i & 0 \\ 0 & 0 \end{bmatrix} \kappa_i \begin{bmatrix} 0 & C_i & F_i & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{K} \end{bmatrix} < 0 \quad (11)$$

where:

$$A_i^0 = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad E_i^0 = \begin{bmatrix} E_i \\ 0 \end{bmatrix}, \quad C_i^0 = \begin{bmatrix} \tilde{C}_i & 0 \end{bmatrix}, \quad F_i^0 = \begin{bmatrix} \tilde{F}_i \end{bmatrix},$$

$$B_i = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad D_i = \begin{bmatrix} I & 0 \end{bmatrix}, \quad C_i = \begin{bmatrix} C_i & 0 \\ 0 & I \end{bmatrix}, \quad F_i = \begin{bmatrix} F_i \end{bmatrix}$$

and $\kappa_i = \begin{bmatrix} D_i^K & C_i^K \\ B_i^K & A_i^K \end{bmatrix}$
Eliminate the $\mathcal{K}$-term

First apply the projection lemma on (11), it yields to the following inequalities:

$$W_{\mathcal{M}_i}^T \Phi_i W_{\mathcal{M}_i} < 0 \quad (12a)$$

$$W_{\mathcal{N}_i}^T \Phi_i W_{\mathcal{N}_i} < 0 \quad (12b)$$
Eliminate the \( K \)-term

First apply the projection lemma on (11), it yields to the following inequalities:

\[
W_{\mathcal{M}_i}^T \Phi_i W_{\mathcal{M}_i} < 0 \quad (12a)
\]
\[
W_{\mathcal{N}_i}^T \Phi_i W_{\mathcal{N}_i} < 0 \quad (12b)
\]

Then compute \( W_{\mathcal{M}_i} \) and \( W_{\mathcal{N}_i} \), and the definition of \( P_i \) the following theorem is deduced

**Theorem (Suboptimal discrete-time fault filter)**

The suboptimal \( H_{\infty} \) fault estimation problem is solvable if and only if there exist positive definite matrices \( R_{i/j} \) and \( S_{i/j} \) \( \forall i, j \in \{1..N\} \) such that:

\[
\begin{bmatrix}
-R_j + A_i R_i A_i^T & E_i \\
E_i^T & -\gamma_i^2 I
\end{bmatrix} < 0 \quad (13a)
\]

\[
\begin{bmatrix}
W_{\mathcal{CF}_i}^T & 0 \\
0 & I_{n_f}
\end{bmatrix}
\begin{bmatrix}
-S_i + A_i^T S_j A_i & A_i^T S_j E_i & \tilde{C}_i^T \\
E_i^T S_j A_i & -\gamma_i^2 I + E_i^T S_j E_i & \tilde{F}_i^T \\
\tilde{C}_i & \tilde{F}_i & -I
\end{bmatrix}
\begin{bmatrix}
W_{\mathcal{CF}_i} & 0 \\
0 & I_{n_f}
\end{bmatrix} < 0 \quad (13b)
\]

Where \( W_{\mathcal{CF}_i} \) are basis of null spaces of \( \begin{bmatrix} C_i & F_i \end{bmatrix} \)
Eliminate the $\mathcal{K}$-term

First apply the projection lemma on (11), it yields to the following inequalities:

\[
W_{\mathcal{M}_i}^T \Phi_i W_{\mathcal{M}_i} < 0 \\
W_{\mathcal{N}_i}^T \Phi_i W_{\mathcal{N}_i} < 0
\] (12a)

Then compute $W_{\mathcal{M}_i}$ and $W_{\mathcal{N}_i}$, and the definition of $P_i$ the following theorem is deduced

Theorem (Suboptimal discrete-time fault filter)

The suboptimal $H_\infty$ fault estimation problem is solvable if and only if there exist positive definite matrices $R_{i/j}$ and $S_{i/j} \forall i, j \in \{1..N\}$ such that:

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E_i^T & -\gamma_i^2 I
\end{bmatrix} < 0 \quad (13a)
\]

\[
\begin{bmatrix}
W_{CF_i}^T & 0 \\
0 & I_{nf}
\end{bmatrix}
\begin{bmatrix}
-S_i + A_i^T S_j A_i & A_i^T S_j E_i & \tilde{C}_i^T \\
E_i^T S_j A_i & -\gamma_i^2 I + E_i^T S_j E_i & \tilde{F}_i \\
\tilde{C}_i^T & \tilde{F}_i & -I
\end{bmatrix}
\begin{bmatrix}
W_{CF_i} \\
0 \\
I_{nf}
\end{bmatrix} < 0 \quad (13b)
\]

Where $W_{CF_i}$ are basis of null spaces of $[C_i \quad F_i]$
Recall $P_i$, defined as:

$$P_i = \begin{bmatrix} S_i & N_i \\ N_i^T & U_i \end{bmatrix} = \begin{bmatrix} R_i & M_i \\ M_i^T & V_i \end{bmatrix}^{-1}$$  \quad (14)
Controller reconstruction: find $P$ (part 1)

Recall $P_i$, defined as:

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$R_i, S_i$ are computed as solution for (13a), we need to find the other elements of $P_i$. 

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Controller reconstruction: find $P$ (part 1)

Recall $P_i$, defined as:

$$P_i = \begin{bmatrix} S_i & N_i \\ N_i^T & U_i \end{bmatrix} = \begin{bmatrix} R_i & M_i \\ M_i^T & V_i \end{bmatrix}^{-1}$$ (14)

$R_i, S_i$ are computed as solution for (13a), we need to find the other elements of $P_i$

From $P_iP_i^{-1} = I$, we infer:

$$P_i \begin{bmatrix} R_i \\ M_i^T \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \iff \begin{bmatrix} S_i & N_i \\ N_i^T & U_i \end{bmatrix} \begin{bmatrix} R_i \\ M_i^T \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$ (15)
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M_i^T & V_i
\end{bmatrix}^{-1}
$$

(14)

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$$

(15)

From (15), the matrices $M_i, N_i, U_i$ can be computed using:

$$
S_i R_i + N_i M_i^T = I 
$$

(16a)

$$
N_i^T R_i + U_i M_i^T = 0
$$

(16b)
Controller reconstruction: find $P$ (part 1)

Recall $P_i$, defined as:

$$P_i = \begin{bmatrix} S_i & N_i \\ N_i^T & U_i \end{bmatrix} = \begin{bmatrix} R_i & M_i \\ M_i^T & V_i \end{bmatrix}^{-1}$$

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(15)

From (15), the matrices $M_i, N_i, U_i$ can be computed using:

$$S_iR_i + N_iM_i^T = I$$

(16a)

$$N_i^T R_i + U_iM_i^T = 0$$

(16b)

Product of two unknown matrices!
Recall $P_i$, defined as:

$$P_i = \begin{bmatrix} S_i & N_i \\ N_i^T & U_i \end{bmatrix} = \begin{bmatrix} R_i & M_i \\ M_i^T & V_i \end{bmatrix}^{-1}$$ (14)

$R_i, S_i$ are computed as solution for (13a), we need to find the other elements of $P_i$

From $P_iP_i^{-1} = I$, we infer:

$$P_i \begin{bmatrix} R_i \\ M_i^T \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} S_i & N_i \\ N_i^T & U_i \end{bmatrix} \begin{bmatrix} R_i \\ M_i^T \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$ (15)

From (15), the matrices $M_i, N_i, U_i$ can be computed using:

$$S_iR_i + N_iM_i^T = I$$ (16a)

$$N_i^T R_i + U_iM_i^T = 0$$ (16b)

Product of two unknown matrices! $\Rightarrow$ Singular value decomposition
Controller reconstruction: find P (part 2)

From (16a): denote $X_i = I - S_i R_i$, the singular value decomposition of $X_i$ is:

$$X_i = \Sigma_i \Lambda_i \Gamma_i^T$$

where $\Sigma_i$ is an unitary orthogonal matrix, $\Lambda_i$ is a diagonal matrix, and $\Gamma_i$ is the transpose of an unitary orthogonal matrix.
Controller reconstruction: find P (part 2)

From (16a): denote $X_i = I - S_i R_i$, the singular value decomposition of $X_i$ is:

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(17)

where $\Sigma_i$ is an unitary orthogonal matrix, $\Lambda_i$ is a diagonal matrix, and $\Gamma_i$ is the transpose of an unitary orthogonal matrix.

We introduce to (17) a non singular matrix $G_i$ of appropriate dimensions,

$$X_i = \Sigma_i \Lambda_i G_i G_i^{-1} \Gamma_i^T$$

(18)
From (16a): denote $X_i = I - S_i R_i$, the singular value decomposition of $X_i$ is:

$$X_i = \Sigma_i \Lambda_i \Gamma_i^T$$  \hspace{1cm} (17)

where $\Sigma_i$ is an unitary orthogonal matrix, $\Lambda_i$ is a diagonal matrix, and $\Gamma_i$ is the transpose of an unitary orthogonal matrix.

We introduce to (17) a non singular matrix $G_i$ of appropriate dimensions,

$$X_i = \Sigma_i \Lambda_i G_i G_i^{-1} \Gamma_i^T$$  \hspace{1cm} (18)

Then a solution of $M_i$ and $N_i$ (16a) is:

$$N_i = \Sigma_i \Lambda_i G_i$$  \hspace{1cm} (19a)

$$M_i^T = G_i^{-1} \Gamma_i^T$$  \hspace{1cm} (19b)

$$U_i = M_i^{-1} R_i N_i$$  \hspace{1cm} (19c)
Controller reconstruction: find $P$ (part 2)

From (16a): denote $X_i = I - S_iR_i$, the singular value decomposition of $X_i$ is:

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where $\Sigma_i$ is an unitary orthogonal matrix, $\Lambda_i$ is a diagonal matrix, and $\Gamma_i$ is the transpose of an unitary orthogonal matrix.

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Then a solution of $M_i$ and $N_i$ (16a) is:

$$N_i = \Sigma_i \Lambda_i G_i$$

(19a)

$$M_i^T = G_i^{-1} \Gamma_i^T$$

(19b)

$$U_i = M_i^{-1} R_i N_i$$

(19c)

**Remark 1: One degree of freedom**

The choice of matrix $G_i$ offers one degree of freedom of the filter design.
Extension for uncertain switched LTI systems
Lemma (Majoration lemma)

If $\Delta^T \Delta < Q_{\Delta}$, then for any $\alpha > 0$:

$$X^T \Delta Y + Y^T \Delta X \leq \alpha X^T X + \frac{1}{\alpha} Y^T Q_{\Delta} Y$$ (20)
Lemma (Majoration lemma)

If $\Delta^T \Delta < Q_\Delta$, then for any $\alpha > 0$:

$$X^T \Delta Y + Y^T \Delta X \leq \alpha X^T X + \frac{1}{\alpha} Y^T Q_\Delta Y \tag{20}$$

Theorem (Bounded real lemma for uncertain switched system)

For a given switched linear uncertain system under arbitrary switching, if there exist matrices $P_i$ and a positive definite matrices $P_i \forall i, j \in \{1..N\}$ such that:

$$
\begin{bmatrix}
-P_i^{-1} & A_i^a & E_i^a & 0 & H_i^a & 0 \\
A_i^{aT} & -P_i & 0 & C_i^{aT} & 0 & N_i^{aT} \\
E_i^{aT} & 0 & -\gamma_i^2 I & F_i^{aT} & 0 & 0 \\
0 & C_i^a & F_i^a & -I & 0 & 0 \\
H_i^{aT} & 0 & 0 & 0 & -\alpha_i I^{-1} & 0 \\
0 & N_i^{aT} & 0 & 0 & 0 & -\alpha_i I \\
\end{bmatrix} < 0 \tag{21}
$$

Then a robust switched discrete time fault estimation filter (RSDTFEF) can be designed where the condition (8) is guaranteed.
Proof of previous theorem

For the uncertain switched system, the BRL gives:

\[
\begin{bmatrix}
-\bar{P}^{-1} & \bar{A}^a_i & E^a_i & 0 \\
\bar{A}^{aT}_i & -P_i & 0 & C^{aT}_i \\
E^{aT}_i & 0 & -\gamma^2_i I & F^{aT}_i \\
0 & C^a_i & F^a_i & -I \\
\end{bmatrix} < 0
\]

(22)
For the uncertain switched system, the BRL gives:

\[
\begin{bmatrix}
-P^{-1} & \tilde{A}_i^a & E_i^a & 0 \\
\tilde{A}_i^{aT} & -P_i & 0 & C_i^{aT} \\
E_i^{aT} & 0 & -\gamma^2_i I & F_i^a \\
0 & C_i^a & F_i^a & -I
\end{bmatrix} < 0
\]  

(22)

With: \( \tilde{A}_i^a = \begin{bmatrix} \tilde{A}_i & 0 \\ B_i^K C_i & A_i^K \end{bmatrix} = \begin{bmatrix} A_i + H_{A,i} \Delta A_i N_{A,i} & 0 \\ B_i^K C_i & A_i^K \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ B_i^K C_i & A_i^K \end{bmatrix} + H_{A,i}^a \Delta A_i N_{A,i}^a, \)

where \( H_{A,i}^a = \begin{bmatrix} H_{A,i} \\ 0 \end{bmatrix} \) and \( N_{A,i}^a = \begin{bmatrix} N_{A,i} \\ 0 \end{bmatrix}. \)
Proof of previous theorem

For the uncertain switched system, the BRL gives:

\[
\begin{bmatrix}
-P_i^{-1} & \bar{A}_i^a & E_i^a & 0 \\
\bar{A}_i^{aT} & -P_i & 0 & C_i^{aT} \\
E_i^{aT} & 0 & -\gamma_i^2 I & F_i^a \\
0 & C_i^a & F_i^a & -I
\end{bmatrix} < 0 \tag{22}
\]

With: 

\[
\bar{A}_i^a = \begin{bmatrix} \bar{A}_i & 0 \\ B_i^K C_i & \bar{A}_i^K \end{bmatrix} = \begin{bmatrix} A_i + H_{A,i} \Delta A_i N_{A,i} & 0 \\ B_i^K C_i & A_i^K \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ B_i^K C_i & A_i^K \end{bmatrix} + H_{A,i} \Delta A_i N_{A,i},
\]

where 

\[
H_{A,i} = \begin{bmatrix} H_{A,i} \\ 0 \end{bmatrix} \quad \text{and} \quad N_{A,i} = \begin{bmatrix} N_{A,i} \\ 0 \end{bmatrix}.
\]

Following the same of calculation as in the previous part, the BMI to be solved is:

\[
\Phi_i + \begin{bmatrix} H_{A,i}^{aT} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \Delta A_i \begin{bmatrix} 0 \\ N_{A,i}^a \\ 0 \\ 0 \end{bmatrix} + [\ast] < 0 \tag{23}
\]
Proof of previous theorem

For the uncertain switched system, the BRL gives:

\[
\begin{bmatrix}
 -P_i^{-1} & \bar{A}_i^a & E_i^a & 0 \\
 \bar{A}_i^{aT} & -P_i & 0 & C_i^a F_i^a \\
 E_i^{aT} & 0 & -\gamma_i^2 I & F_i^{aT} \\
 0 & C_i^a & F_i^a & -I
\end{bmatrix} < 0 \tag{22}
\]

With: 
\[
\bar{A}_i^a = \begin{bmatrix} \bar{A}_i & 0 \\ B_i^K C_i & A_i^K \end{bmatrix} = \begin{bmatrix} A_i + H_{A,i} \Delta A_i, i N_{A,i} & 0 \\ B_i^K C_i & A_i^K \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ B_i^K C_i & A_i^K \end{bmatrix} + \begin{bmatrix} H_{A,i} \Delta A_i, i N_{A,i} \\ 0 \end{bmatrix},
\]

where 
\[
H_{A,i} = \begin{bmatrix} H_{A,i} \\ 0 \end{bmatrix} \quad \text{and} \quad N_{A,i} = \begin{bmatrix} N_{A,i} & 0 \end{bmatrix}.
\]

Following the same of calculation as in the previous part, the BMI to be solved is:

\[
\Phi_i + \begin{bmatrix} H_{A_i}^{aT} & 0 & 0 & 0 \end{bmatrix}^T \Delta_{A,i} \begin{bmatrix} 0 & N_{A_i} & 0 & 0 \end{bmatrix} + [\ast] < 0 \tag{23}
\]

Apply Lemma 4 with 
\[
X = \begin{bmatrix} H_{A_i}^{aT} & 0 & 0 & 0 \end{bmatrix}^T, \quad Y = \begin{bmatrix} 0 & N_{A_i} & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q_\Delta = I.
\]
Proof of previous theorem

For the uncertain switched system, the BRL gives:

\[
\begin{bmatrix}
-P_i^{-1} & \bar{A}_i^a & E_i^a & 0 \\
\bar{A}_i^{aT} & -P_i & 0 & C_i^{aT} \\
E_i^{aT} & 0 & -\gamma_i^2 I & F_i^a \\
0 & C_i^a & F_i^a & -I
\end{bmatrix} < 0 \quad (22)
\]

With: \( \bar{A}_i^a = \begin{bmatrix} \bar{A}_i \\ B_i^K C_i \\ A_i^K \end{bmatrix} = \begin{bmatrix} A_i + H_{A,i} \Delta_{A,i} N_{A,i} \\ B_i^K C_i \\ A_i^K \end{bmatrix} = \begin{bmatrix} A_i \\ B_i^K C_i \\ A_i^K \end{bmatrix} + H_{A,i} \Delta_{A,i} N_{A,i}^a\),

where \( H_{A,i} = \begin{bmatrix} H_{A,i} \\ 0 \end{bmatrix} \) and \( N_{A,i}^a = \begin{bmatrix} N_{A,i} \\ 0 \end{bmatrix} \).

Following the same of calculation as in the previous part, the BMI to be solved is:

\[
\Phi_i + \begin{bmatrix} H_{A,i}^{aT} & 0 & 0 & 0 \end{bmatrix}^T \Delta_{A,i} \begin{bmatrix} 0 & N_{A,i}^a & 0 & 0 \end{bmatrix} + [*] < 0 \quad (23)
\]

Apply Lemma 4 with \( X = \begin{bmatrix} H_{A,i}^{aT} & 0 & 0 & 0 \end{bmatrix}^T \), \( Y = \begin{bmatrix} 0 & N_{A,i}^a & 0 & 0 \end{bmatrix} \) and \( Q_\Delta = I \).

And finally with two Shur complement, the BMI (21) is deduced.
The suboptimal $H_{\infty}$ fault estimation problem is solvable if and only if there exist positive definite matrices $R_i$ and $S_i \ \forall i \in \{1..N\}$ such that:

\[
\begin{bmatrix}
-R_j + A_i R_i A_i^T & E_i & H_{A,i} \\
E_i^T & -\gamma_i^2 I & 0 \\
H_{A,i}^T & 0 & -\alpha_i^{-1} I \\
N_{A,i} R_i A_i^T & 0 & 0 \\
\end{bmatrix} < 0
\]

(24a)

\[
\mathcal{W}_{CF_i}^T \begin{bmatrix}
A_i^T S_j A_i - S_i \\
E_i^T S_j E_i \\
\tilde{C}_i \\
H_{A,i}^T S_j A_i \\
N_{A,i}
\end{bmatrix} \begin{bmatrix}
A_i^T S_j E_i \\
E_i^T S_j E_i - \gamma_i^2 I \\
\tilde{F}_i \\
H_{A,i}^T S_j E_i \\
0
\end{bmatrix} \begin{bmatrix}
\tilde{C}_i \\
\tilde{F}_i \\
-I \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
A_i^T S_j H_{A,i} \\
E_i^T S_j H_{A,i} \\
0 \\
0 \\
0 \\
\end{bmatrix} \begin{bmatrix}
N_{A,i}^T \\
0 \\
0 \\
0 \\
-\alpha_i I
\end{bmatrix} < 0
\]

(24b)

Where $\mathcal{W}_{CF_i} = \begin{bmatrix} W_{CF_i} & 0 \\ 0 & I_{n_f+n_h+n_a} \end{bmatrix}$, $\alpha_i$ are free design scalars and $W_{CF_i}$ are basis of null spaces of $[C_i \ F_i]$. 
Improved Robust fault estimation

Dynamical filters can be introduced in the design procedure:

- The filter $W_d(q^{-1})$ imposes robustness toward the disturbances in a specified frequency ranges
- and $W_f(q^{-1})$ is introduced to shape the desired response of $\hat{f}_k$ to the fault.

**Figure:** Fault estimation schemes
Improved Robust fault estimation

Dynamical filters can be introduced in the design procedure:

- The filter $W_d(q^{-1})$ imposes robustness toward the disturbances in a specified frequency ranges
- and $W_f(q^{-1})$ is introduced to shape the desired response of $\hat{f}_k$ to the fault.

The dynamical filters $W_{d,i}$ and $W_{f,i}$ can be defined as switched systems driven by the same switching signal of the system, they have the realization:

$$W_{f,i}(q^{-1}): \begin{cases} x_{k+1}^F &= A_i^F x_k^F + B_i^F f_k \\ f_k &= C_i^F x_k^F + D_i^F f_k \end{cases} \quad (25)$$

$$W_{d,i}(q^{-1}): \begin{cases} x_{k+1}^D &= A_i^D x_k^D + B_i^D d_k \\ d_k &= C_i^D x_k^D + D_i^D d_k \end{cases} \quad (26)$$

where $x_k^F \in \mathbb{R}^{nF}$ and $x_k^D \in \mathbb{R}^{nD}$, and it is assumed that $\text{dim}(f_k) = \text{dim}(\bar{f}_k)$ and $\text{dim}(d_k) = \text{dim}(\bar{d}_k)$.
Augmented system with weighting filters

The new state of the augmented system is now defined by

\[
x_a^k = [x_k^T \ x_k^{FT} \ x_k^{DT} \ x_k^{KT}]^T:
\]

\[
\begin{align*}
x_{k+1}^a &= A_i^a x_{k+1}^a + E_i^a w_k \\
z_k &= C_i^a x_k^a + F_i^a w_k
\end{align*}
\] (27)
Augmented system with weighting filters

The new state of the augmented system is now defined by

$$x_k^a = [x_k^T \ x_k^{FT} \ x_k^{DT} \ x_k^{KT}]^T$$:

$$\begin{align*}
    x_{k+1}^a &= A_i^a x_{k+1}^a + E_i^a w_k \\
    z_k &= C_i^a x_k^a + F_i^a w_k
\end{align*}$$

(27)

Where

$$A_i^a = \begin{bmatrix} \tilde{A}_i & 0 \\ B_i^K \tilde{C}_i & A_i^K \end{bmatrix}, \ E_i^a = \begin{bmatrix} \tilde{E}_{w,i} \\ B_i^K \tilde{F}_{w,i} \end{bmatrix}, \ C_i^a = \begin{bmatrix} D_i^K \tilde{C}_i + \tilde{C}_i \\ C_i^K \end{bmatrix},$$

$$F_i^a = D_i^K \tilde{F}_{w,i} + \tilde{F}_i, \ \tilde{A}_i = \begin{bmatrix} A_i & 0 & E_{d,i} C_i^D \\ 0 & A_i^F & 0 \\ 0 & 0 & A_i^D \end{bmatrix}, \ \tilde{E}_{w,i} = \begin{bmatrix} E_{d,i} D_i^D & E_{f,i} \\ 0 & B_i^F \\ B_i^D & 0 \end{bmatrix},$$

$$\tilde{C}_i = \begin{bmatrix} C_i & 0 & F_{d,i} C_i^D \\ 0 & 0 & A_i^D \end{bmatrix}, \ \tilde{F}_{w,i} = \begin{bmatrix} F_{d,i} D_i^D & F_{f,i} \end{bmatrix}, \ \tilde{C}_i = \begin{bmatrix} 0 & -C_i^F & 0 \end{bmatrix}, \ \text{and} \ \tilde{F}_i = \begin{bmatrix} 0 & -\tilde{D}_i^F \end{bmatrix}.$$
Augmented system with weighting filters

The new state of the augmented system is now defined by
\[
x^a_k = [x^T_k x^{FT}_k x^{DT}_k x^{KT}_k]^T:
\]
\[
\begin{cases}
    x^a_{k+1} = A^a x^a_k + E^a w_k \\
    z_k = C^a x^a_k + F^a w_k
\end{cases}
\]

Where \( A^a_i = \begin{bmatrix} \tilde{A}_i & 0 \\ B^K_i \tilde{C}_i & A^K_i \end{bmatrix} \), \( E^a_i = \begin{bmatrix} \tilde{E}_{w,i} \\ B^K_i \tilde{F}_{w,i} \end{bmatrix} \), \( C^a_i = \begin{bmatrix} D^K_i \tilde{C}_i + \tilde{C}_i & C^K_i \end{bmatrix} \),
\[
F^a_i = D^K_i \tilde{F}_{w,i} + \tilde{F}_i, \quad \tilde{A}_i = \begin{bmatrix} A_i & 0 & E_{d,i} C^D_i \\ 0 & A^F_i & 0 \\ 0 & 0 & A^D_i \end{bmatrix}, \quad \tilde{E}_{w,i} = \begin{bmatrix} E_{d,i} D^D_i & E_{f,i} \\ 0 & B^F_i \\ B^D_i & 0 \end{bmatrix},
\]
\[
\tilde{C}_i = \begin{bmatrix} C_i & 0 & F_{d,i} C^D_i \end{bmatrix}, \quad \tilde{F}_{w,i} = \begin{bmatrix} F_{d,i} D^D_i & F_{f,i} \end{bmatrix}, \quad \tilde{C}_i = \begin{bmatrix} 0 & -C^F_i & 0 \end{bmatrix}, \quad \text{and}
\]
\[
\tilde{F}_i = \begin{bmatrix} 0 & -\tilde{D}_i^F \end{bmatrix}.
\]

Using the new notation, theorem 2 can be applied on the augmented system.
Augmented system with weighting filters

The new state of the augmented system is now defined by

\[
x^a_k = \begin{bmatrix} x_k^T & x_k^{FT} & x_k^{DT} & x_k^{KT} \end{bmatrix}^T:
\]

\[
\left\{ \begin{array}{l}
x^a_{k+1} = A^a_i x^a_k + E^a_i w_k \\
z_k = C^a_i x^a_k + F^a_i w_k
\end{array} \right.
\]

Where

\[
A^a_i = \begin{bmatrix} \tilde{A}_i & 0 \\ B^K_i \tilde{C}_i & A^K_i \end{bmatrix},
E^a_i = \begin{bmatrix} \tilde{E}_{w,i} \\ B^K_i \tilde{F}_{w,i} \end{bmatrix},
C^a_i = \begin{bmatrix} D^K_i \tilde{C}_i + \tilde{C}_i & C^K_i \end{bmatrix},
\]

\[
F^a_i = D^K_i \tilde{F}_{w,i} + \tilde{F}_i,
\tilde{A}_i = \begin{bmatrix} A_i & 0 & E_{d,i} C^D_i \\ 0 & A^F_i & 0 \\ 0 & 0 & A^D_i \end{bmatrix},
\tilde{E}_{w,i} = \begin{bmatrix} E_{d,i} D^D_i & E_{f,i} \\ 0 & B^F_i \\ B^D_i & 0 \end{bmatrix},
\tilde{C}_i = \begin{bmatrix} C_i & 0 & F_{d,i} C^D_i \end{bmatrix},
\tilde{F}_{w,i} = \begin{bmatrix} F_{d,i} D^D_i & F_{f,i} \end{bmatrix},
\tilde{C}_i = \begin{bmatrix} 0 & -C^F_i & 0 \end{bmatrix}, \quad \text{and}
\tilde{F}_i = \begin{bmatrix} 0 & -\tilde{D}^F_i \end{bmatrix}.
\]

Using the new notation, theorem 2 can be applied on the augmented system.

**Remark 2: Post scaling for low frequencies estimation**

In order to get an exact estimate of the fault signal in low frequency range, an a posteriori scaling factor can be added on the output of the \( K_i \) filter.
Consider the uncertain switched LTI system:

\[
\begin{aligned}
    x_{k+1} &= (A_\alpha(k) + H_{A,\alpha(k)} \Delta_{A,\alpha(k)} N_{A,\alpha(k)}) x_k + E_{d,\alpha(k)} d_k + E_{f,\alpha(k)} f_k \\
    y_k &= C_\alpha(k) x_k + F_{d,\alpha(k)} d_k + F_{f,\alpha(k)} f_k
\end{aligned}
\]  

(28)

with the following nominal matrices:

\[
A_1 = \begin{bmatrix} 1 & 0.1 \\ -0.1 & 0.89 \end{bmatrix},
A_2 = \begin{bmatrix} 1 & 0.07 \\ 0.1 & 0.91 \end{bmatrix},
E_{d,1} = E_{d,2} = \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix},
E_{f,1} = \begin{bmatrix} 0.02 \\ 0.01 \end{bmatrix},
E_{f,2} = \begin{bmatrix} 0.022 \\ 0.011 \end{bmatrix},
C_1 = C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},
F_{d,1} = F_{d,2} = \begin{bmatrix} 0.01 \end{bmatrix},
F_{f,1} = F_{f,2} = [1.6].
\]

And for the uncertainties directions:

\[
H_{A,1} = H_{A,2} = \begin{bmatrix} 0.2 & 0.002 \end{bmatrix}^T, 
N_{A,1} = \begin{bmatrix} 0.22 & .128 \end{bmatrix} \text{ and } N_{A,2} = \begin{bmatrix} 0.22 & .13 \end{bmatrix}
\]
Numerical example for two design methods: design procedure

The system is observable, and the matrices dimensions meet the assumptions. A RSDTFEF can be designed, using two previous methods.
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Numerical example for two design methods: design procedure

The system is observable, and the matrices dimensions meet the assumptions. A RSDTFEF can be designed, using two previous methods.

Figure: First approach: Fault estimation filter
Numerical example for two design methods: design procedure

The system is observable, and the matrices dimensions meet the assumptions. A RSDTFEF can be designed, using two previous methods.

First, define the loop shaping matrices and weighting filter $W_f$ and $W_d$. 

**Figure:** First approach: Fault estimation filter  

**Figure:** Second approach: improved filtering
The system is observable, and the matrices dimensions meet the assumptions. A RSDTFEF can be designed, using two previous methods.

First, define the loop shaping matrices and weighting filter $W_f$ and $W_d$.

Then using Matlab optimization tools such YALMIP or SeDuMi, the set of LMIs are solved minimizing the criterion $\alpha_i$. 

**Figure:** First approach: Fault estimation filter

**Figure:** Second approach: improved filtering
Numerical example for two design methods: design procedure

The system is observable, and the matrices dimensions meet the assumptions. A RSDTFEF can be designed, using two previous methods.

First, define the loop shaping matrices and weighting filter $W_f$ and $W_d$.

Then using Matlab optimization tools such YALMIP or SeDuMi, the set of LMIs are solved minimizing the criterion $\alpha_i$.

The post filters scheduling factors $\xi_i$ can be calculated and added to $K_i$ according to remark 2: $\tilde{K}_i(q^{-1}) = \xi_i K_i(q^{-1})$
Numerical example for two design methods: results

In this example:
- The considered perturbation is a white noise.
- two fault signals are considered: abrupt fault and sinusoidal one.
Numerical example for two design methods: results

In this example:

- The considered perturbation is a white noise.
- Two fault signals are considered: abrupt fault and sinusoidal one.

Figure: Switching sequence

Figure: Disturbance signal
Numerical example for two design methods: results

In this example:

- The considered perturbation is a white noise.
- two fault signals are considered: abrupt fault and sinusoidal one.

The two approaches for filter design are implemented in order to estimate the fault $f_k$. 

**Figure:** Switching sequence

**Figure:** disturbance signal
Numerical example for two design methods: results

In this example:

- The considered perturbation is a white noise.
- Two fault signals are considered: abrupt fault and sinusoidal one.

The two approaches for filter design are implemented in order to estimate the fault $f_k$.

Figure: Switching sequence

Figure: Disturbance signal

Figure: Fault estimation without filtering
In this example:

- The considered perturbation is a white noise.
- Two fault signals are considered: abrupt fault and sinusoidal one.

![Switching sequence](image1)

**Figure:** Switching sequence

![Disturbance signal](image2)

**Figure:** Disturbance signal

The two approaches for filter design are implemented in order to estimate the fault $f_k$.

![Fault estimation without filtering](image3)

**Figure:** Fault estimation without filtering

![Fault estimation with filtering](image4)

**Figure:** Fault estimation with filtering
Vehicle fault detection
Test campaign

- Collaboration with MIPS (Mulhouse), CAOR (Mines Paris-tech) and SOBEN, in the ANR Project INOVE.
- Test campaign on instrumented Renault car with professional pilot.
Test campaign

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- Test campaign on instrumented Renault car with professional pilot.
Consider the non linear bicycle model of the vehicle for lateral control:

\[
\begin{bmatrix}
\dot{\beta}(t) \\
\dot{\psi}(t)
\end{bmatrix} = \begin{bmatrix}
\frac{-c_r + c_f}{mv(t)} & \frac{c_r l_r - c_f l_f}{mv^2(t)} - 1 \\
\frac{c_r l_r - c_f l_f}{l_z} & -\frac{c_r l_r^2 + c_f l_f^2}{l_z v(t)}
\end{bmatrix} \begin{bmatrix}
\beta(t) \\
\dot{\psi}(t)
\end{bmatrix} + \begin{bmatrix}
\frac{c_f}{mv(t)} \\
\frac{1}{l_w l_z}
\end{bmatrix} u_L(t) + \begin{bmatrix}
\frac{1}{mv(t)} \\
\frac{1}{l_w l_z}
\end{bmatrix} F_w(t)
\]

\( y = \begin{bmatrix}
-\frac{c_r + c_f}{m} & c_f l_f - c_r l_r
\end{bmatrix} \begin{bmatrix}
\beta(t) \\
\dot{\psi}(t)
\end{bmatrix} + \begin{bmatrix}
\frac{c_f}{m}
\end{bmatrix} u_L(t) \)  

- states: side slip angle \( \beta \) and the yaw rate \( \dot{\psi} \)
- command: the steering angle \( u_L \)
- \( y \): lateral acceleration \( \gamma_L \)
- perturbation: wind force \( F_w \)
Consider the non-linear bicycle model of the vehicle for lateral control:

\[
\begin{bmatrix}
\dot{\beta}(t) \\
\dot{\psi}(t)
\end{bmatrix} = \begin{bmatrix}
\frac{-c_r + c_f}{m v(t)} & \frac{c_r l_f - c_f l_f}{m v^2(t)} - 1 \\
\frac{c_r l_r - c_f l_f}{l_z} & \frac{-c_r l_f^2 + c_f l_f^2}{l_z v(t)}
\end{bmatrix} \begin{bmatrix}
\beta(t) \\
\dot{\psi}(t)
\end{bmatrix} + \begin{bmatrix}
c_f \\
\frac{1}{m v(t)}
\end{bmatrix} u_L(t) + \begin{bmatrix}
\frac{1}{l_w} \\
\frac{1}{l_z}
\end{bmatrix} F_w(t)
\]

\[y = \begin{bmatrix}
\dot{\beta}(t) \\
\dot{\psi}(t)
\end{bmatrix} + \begin{bmatrix}
c_f \\
\frac{c_f}{m}
\end{bmatrix} u_L(t)\]

- states: side slip angle \(\beta\) and the yaw rate \(\dot{\psi}\)
- command: the steering angle \(u_L\)
- \(y\): lateral acceleration \(\gamma_L\)
- perturbation: wind force \(F_w\)

The fault considered in this application is an actuator fault, that occurs on the actuator.
Consider the non-linear bicycle model of the vehicle for lateral control:

\[
\begin{bmatrix}
\dot{\beta}(t) \\
\dot{\psi}(t)
\end{bmatrix} = \begin{bmatrix}
\frac{c_r + c_f}{m v(t)} & \frac{c_r l_r - c_f l_f}{l_z} \\
\frac{c_r l_r - c_f l_f}{l_z} & \frac{c_r l_r^2 + c_f l_f^2}{l_z v(t)} - 1
\end{bmatrix}
\begin{bmatrix}
\beta(t) \\
\dot{\psi}(t)
\end{bmatrix} + \begin{bmatrix}
\frac{c_f}{m v(t)} \\
\frac{c_f}{c_r l_f}
\end{bmatrix} u_L(t) + \begin{bmatrix}
\frac{1}{l_w(t)}
\end{bmatrix} F_w(t)
\]

\[
y = \begin{bmatrix}
\frac{c_r + c_f}{m} & c_f l_f - c_r l_f
\end{bmatrix}
\begin{bmatrix}
\beta(t) \\
\dot{\psi}(t)
\end{bmatrix} + \begin{bmatrix}
\frac{c_f}{m}
\end{bmatrix} u_L(t)
\]

\text{(29)}

- states: side slip angle $\beta$ and the yaw rate $\dot{\psi}$
- command: the steering angle $u_L$
- $y$: lateral acceleration $\gamma_L$
- perturbation: wind force $F_w$

The fault considered in this application is an actuator fault, that occurs on the actuator.

Using a Taylor expansion around the points $v_\alpha$:

\[
\frac{1}{v_\alpha} \bigg|_{v = v_\alpha} = \frac{1}{v_\alpha} - \frac{1}{v_\alpha^2} (v - v_\alpha) + \mathcal{O}\left(\frac{1}{v^2}\right)
\]

\[
\frac{1}{v_\alpha^2} \bigg|_{v = v_\alpha} = \frac{1}{v_\alpha^2} - \frac{2}{v_\alpha^3} (v - v_\alpha) + \mathcal{O}\left(\frac{1}{v^3}\right)
\]

\text{(30)}

\text{(31)}

Then

\[
A = A_0 + \frac{1}{v_\alpha} A_1 + \frac{1}{v_\alpha^2} A_2 + \left( - \frac{1}{v_\alpha^2} A_1 - \frac{2}{v_\alpha^3} A_2 \right) (v - v_\alpha)
\]

\[
A_\alpha \quad H_{A_\alpha} \quad \Delta x_\alpha
\]
Consider the scenario of moose test (avoidance test):
Consider the scenario of moose test (avoidance test):

**Figure:** Steering angle [°]
Consider the scenario of moose test (avoidance test):

**Figure:** Steering angle [°]

**Figure:** Longitudinal velocity [km/h] and switching rule
Consider the scenario of moose test (avoidance test):

**Figure:** Steering angle [°]

**Figure:** Longitudinal velocity [km/h] and switching rule

**Figure:** Lateral acceleration [m/s²]
The vehicle model can be put on the form (using the Taylor expansion):

\[
\begin{align*}
\mathbf{x}_{k+1} &= \left( A_{\alpha(k)} + H_{\alpha(k)} \Delta_{\alpha(k)} N_{A,\alpha(k)} \right) \mathbf{x}_k \\
&\quad + \left( E_{d,\alpha(k)} + H_{D,\alpha(k)} \Delta_{\alpha(k)} N_{D,\alpha(k)} \right) d_k \\
&\quad + \left( E_{f,\alpha(k)} + H_{F,\alpha(k)} \Delta_{\alpha(k)} N_{F,\alpha(k)} \right) f_k \\
\mathbf{y}_k &= C_{\alpha(k)} \mathbf{x}_k + F_{d,\alpha(k)} d_k + F_{f,\alpha(k)} f_k
\end{align*}
\]

(32)

Where \( d_k = \begin{bmatrix} u_L(k) & F_w(k) \end{bmatrix} \) as unknown inputs.
The vehicle model can be put on the form (using the Taylor expansion):

\[
\begin{align*}
x_{k+1} &= (A_{\alpha(k)} + H_{A,\alpha(k)} \Delta_{\alpha(k)} N_{A,\alpha(k)}) x_k \\
&\quad + (E_{d,\alpha(k)} + H_{D,\alpha(k)} \Delta_{\alpha(k)} N_{D,\alpha(k)}) d_k \\
&\quad + (E_{f,\alpha(k)} + H_{F,\alpha(k)} \Delta_{\alpha(k)} N_{F,\alpha(k)}) f_k \\
y_k &= C_{\alpha(k)} x_k + F_{d,\alpha(k)} d_k + F_{f,\alpha(k)} f_k
\end{align*}
\]  
(32)

Where \( d_k = [u_L(k) \quad F_w(k)] \) as unknown inputs

and the uncertainties matrices:

\[
H_{\alpha(k)} \Delta_{\alpha(k)} N_{\alpha(k)} = 
\begin{bmatrix}
H_{A,\alpha(k)} & H_{D,\alpha(k)} & H_{F,\alpha(k)}
\end{bmatrix} \Delta_{\alpha(k)} 
\begin{bmatrix}
N_{A,\alpha(k)} & N_{D,\alpha(k)} & N_{F,\alpha(k)}
\end{bmatrix}
\]
The vehicle model can be put on the form (using the Taylor expansion):

\[
\begin{align*}
    x_{k+1} &= \left(A_{\alpha(k)} + H_{A,\alpha(k)} \Delta_{\alpha(k)} N_{A,\alpha(k)}\right)x_k \\
    &\quad + \left(E_{d,\alpha(k)} + H_{D,\alpha(k)} \Delta_{\alpha(k)} N_{D,\alpha(k)}\right)d_k \\
    &\quad + \left(E_{f,\alpha(k)} + H_{F,\alpha(k)} \Delta_{\alpha(k)} N_{F,\alpha(k)}\right)f_k \\
    y_k &= C_{\alpha(k)} x_k + F_{d,\alpha(k)} d_k + F_{f,\alpha(k)} f_k
\end{align*}
\]  

(32)

Where \( d_k = \begin{bmatrix} u_L(k) & F_w(k) \end{bmatrix} \) as unknown inputs

and the uncertainties matrices:

\[
H_{\alpha(k)} \Delta_{\alpha(k)} N_{\alpha(k)} = \begin{bmatrix} H_{A,\alpha(k)} & H_{D,\alpha(k)} & H_{F,\alpha(k)} \end{bmatrix} \Delta_{\alpha(k)} \begin{bmatrix} N_{A,\alpha(k)} & N_{D,\alpha(k)} & N_{F,\alpha(k)} \end{bmatrix}
\]

The extension is easy for the uncertainties.
The vehicle model can be put on the form (using the Taylor expansion):

\[
\begin{align*}
    x_{k+1} &= (A_{\alpha(k)} + H_{A,\alpha(k)} \Delta_{\alpha(k)} N_{A,\alpha(k)}) x_k \\
             &\quad + (E_{d,\alpha(k)} + H_{D,\alpha(k)} \Delta_{\alpha(k)} N_{D,\alpha(k)}) d_k \\
             &\quad + (E_{f,\alpha(k)} + H_{F,\alpha(k)} \Delta_{\alpha(k)} N_{F,\alpha(k)}) f_k \\
    y_k &= C_{\alpha(k)} x_k + F_{d,\alpha(k)} d_k + F_{f,\alpha(k)} f_k
\end{align*}
\]

(32)

Where \( d_k = [u_L(k) \quad F_w(k)] \) as unknown inputs

and the uncertainties matrices:

\[
H_{\alpha(k)} \Delta_{\alpha(k)} N_{\alpha(k)} = [H_{A,\alpha(k)} \quad H_{D,\alpha(k)} \quad H_{F,\alpha(k)}] \Delta_{\alpha(k)} [N_{A,\alpha(k)} \quad N_{D,\alpha(k)} \quad N_{F,\alpha(k)}]
\]

The extension is easy for the uncertainties. Work in progress.
Conclusions:

-Introduced a design of discrete time fault detection filter for switched system using two approaches, with some degree of freedom
-Extended this method for uncertain switched system
-Illustrated these two approaches with numerical example

Future Work:

-This work is under submission to a journal (IEEE TAC)
-Extension for fault tolerant control strategies in both vertical and lateral vehicle dynamics
-Generalization for uncertain LPV systems, and uncertain switched system, and maybe descriptor uncertain switched system.
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Thank you for your attention

Any questions?